Transactions Letters

Viterbi Decoding of Convolutional Codes in Symmetric α-Stable Noise

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***Abstract*—Algorithms developed with a Gaussian noise assumption perform poorly in impulsive noise, such as that described by the symmetric α-stable (SαS) distribution. We investigate the performance of antipodal signaling and Viterbi decoding of convolutional codes in SαS noise. We demonstrate that the *p*-norm branch metric is robust in SαS noise.**

***Index Terms*—Convolutional codes, impulse noise, Viterbi decoding.**

I. Introduction

THE USE of Gaussian noise assumption in communication systems is justified by the *central limit theorem* and is further motivated by the mathematically tractable probability density function (pdf). However many communication environments do not satisfy the Gaussian noise assumption [1]–[3]. Impulsive noise models such as the Middleton noise model and *stable* distributions [1], [3] have been used to model non-Gaussian noise. The theoretical justification for the use of the stable family of distributions comes from the *generalized central limit theorem* [3]. In this letter, we analyze the performance of uncoded and coded communications in the presence of stable noise.

II. Noise Model

The symmetric α-stable (SαS) distribution is a stable distribution with no skew and zero mean [1], [3]. The SαS distribution is parameterized by the *characteristic exponent* $\alpha (0 < \alpha \leq 2)$ and *dispersion* $\gamma (\gamma > 0)$. The value of α controls the heaviness of the tails; small values of α result in heavy tails while $\alpha = 2$ results in a Gaussian distribution. The scale parameter γ is similar to variance in the Gaussian distribution. All stable distributions with $\alpha < 2$ have an infinite variance but finite dispersion. When $\alpha = 2$, the variance is finite and is equal to 2γ. As most practical environments have noise with α in the range of 1.5–2, we limit our analysis in this paper to $1 < \alpha \leq 2$. The SαS distribution does not have a general closed-form pdf $f_\alpha(x;\gamma)$ or cumulative distribution function (cdf) $F_\alpha(x,\gamma)$ except in the special cases of $\alpha = 1$ and $\alpha = 2$. For other values of α, $f_\alpha(x;\gamma)$ and $F_\alpha(x;\gamma)$ can be evaluated based on efficient numerical approximations [4].

The pdf of the Gaussian distribution ($\alpha = 2$) is known [3], but the cdf is not known in closed form. However, the cdf is well studied and tabulated. It is often used in communication theory in its complementary form as the Q-function:

$$Q(x) = 1 - F_2\left(x;\frac{1}{2}\right) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp\left(-\frac{t^2}{2}\right)dt. \quad (1)$$

The Cauchy distribution is obtained from the SαS distribution when $\alpha = 1$. The pdf and the cdf of the Cauchy distribution are known in closed form [3]. In line with the commonly used Q-function defined in (1), we define a right tail probability function $Q_\alpha(x)$ for the SαS distribution.

$$Q_\alpha(x) = \int_x^\infty f_\alpha(t;1)\,dt. \quad (2)$$

$Q_\alpha(x)$ is not known in closed form. However, the asymptotic behavior of the right tail probability is given by the algebraic tail behavior of stable distributions [3]:

$$\lim_{x\to\infty} Q_\alpha(x) \sim \frac{C_\alpha}{2}x^{-\alpha}$$

where

$$C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, \qquad 1 < \alpha \leq 2. \quad (3)$$

III. Antipodal Signaling Performance

We consider a channel that corrupts a signal only via additive SαS noise. If signal x_t is passed through the channel, we receive y_t such that $y_t = x_t + n_t$, where n_t is a sequence of independent SαS noise samples. In line with an additive white Gaussian noise (AWGN) channel, we call this channel an additive white SαS noise (AWSαSN) channel.

The performance of a communication system can be measured in terms of the signal-to-noise power ratio (SNR) or the ratio of signal energy per bit (E_b) to the noise power spectral density (N_0). Both definitions are in terms of noise power, a quantity that is related to the variance of the noise. As the variance of SαS noise with $\alpha < 2$ is infinite, this definition has to be modified for performance analysis in AWSαSN channel. The signal-to-dispersion ratio has been used previously as a measure of SNR [1]. As the units of dispersion depend on the value of α, we modify this measure by defining N_0 in terms of the dispersion γ such that $N_0 = 4\gamma^{2/\alpha}$. This definition leads to a dimensionless ratio E_b/N_0, with N_0 proportional to the square

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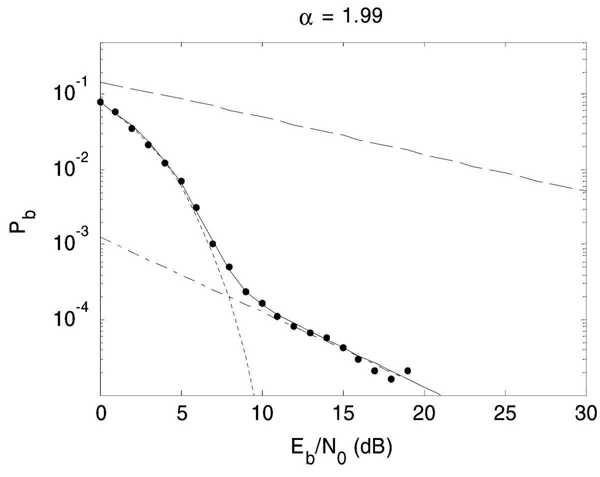

Fig. 1. Performance of antipodal signaling in AWSαSN–based on Q_α (solid line), Cauchy upper bound (dashed line), asymptotic approximation (dash-dot line), performance in Gaussian noise (dotted line), and simulation results (solid dots).

of the scale factor ($\gamma^{1/\alpha}$) of the noise distribution as one would intuitively expect. The factor of 4 is chosen to ensure that this definition reduces to the standard definition of N_0 in the case of Gaussian noise ($\alpha = 2$), making our analysis consistent with previous literature.

We assume that the communication system uses antipodal signaling (such as BPSK) and that bits 0 and 1 are transmitted with equal probability. As shown in the case of the performance of BPSK in AWGN [5], the probability of bit error (P_b) may be expressed in terms of the tail probability of the noise:

$$P_b = Q_\alpha\left(\frac{\sqrt{E_b}}{\gamma^{1/\alpha}}\right) = Q_\alpha\left(\frac{\sqrt{E_b}}{\sqrt{N_0}/2}\right) = Q_\alpha\left(2\sqrt{\frac{E_b}{N_0}}\right). \tag{4}$$

By numerically computing P_b for various values of α, we find that for $E_b/N_0 > 0.25$ (i.e., -6 dB), P_b is upper bounded by the tail probability of the impulsive Cauchy distribution. For large E_b/N_0, the performance can be approximated using the asymptotic tail probability. Thus, we have

$$P_b < \frac{1}{2} - \frac{1}{\pi}\arctan\left(2\sqrt{\frac{E_b}{N_0}}\right), \quad \text{if} \quad \frac{E_b}{N_0} \geq \frac{1}{4} \tag{5}$$

$$P_b \sim \frac{C_\alpha}{2}\left(2\sqrt{\frac{E_b}{N_0}}\right)^{-\alpha} = \frac{C_\alpha}{2^{1+\alpha}}\left(\frac{E_b}{N_0}\right)^{-\alpha/2}$$
$$\text{as} \quad \frac{E_b}{N_0} \to \infty. \tag{6}$$

The numerical approximation of (4), the Cauchy upper bound (5), the asymptotic approximation (6), and the AWGN performance are compared against simulation results in Fig. 1. As one would expect, the performance of AWSαSN is similar to that of AWGN at low E_b/N_0 when the deviation from Gaussian noise is small. However, the performance is significantly poorer at high E_b/N_0 even for a small deviation from Gaussian noise ($\alpha = 1.99$), as the errors are dominated by the tail behavior of the noise distribution. At high E_b/N_0, the asymptotic approximation is quite accurate. The Cauchy upper bound is quite loose at high E_b/N_0, especially when the value of α is also high.

IV. Viterbi Decoding of Convolutional Codes

The Viterbi decoding algorithm provides an optimal and efficient algorithm to decode convolutional codes [6], [7]. The exact probability of bit error for a coded communication system with Viterbi decoding is difficult to determine, but an upper bound on the bit error probability can be obtained in terms of the information of error weight coefficients c_d [7]

$$P_b < \sum_{d=d_f}^{\infty} c_d p_d \tag{7}$$

where d_f is the free distance of the code and p_d is the pair-wise probability of error with weight d. Although the summation in (7) has an infinite number of terms, typically the terms are decreasing in magnitude and the summation of the first few terms provides an acceptable upper bound.

In a hard decision decoded communication system, decisions based on the detector output are fed to the Viterbi decoder, which selects the most likely transmitted data by minimizing the Hamming distance between the received data and all possible transmitted code words. Hard decision decoding of a rate R code effectively converts the AWSαSN channel into a binary symmetric channel (BSC). The pair-wise probability of error for a BSC is upper bounded by $(2\sqrt{\varepsilon(1-\varepsilon)})^d$, where ε is the transition probability of the BSC [6], [7]. Thus,

$$P_b < \sum_{d=d_f}^{\infty} c_d \times \left(2\sqrt{Q_\alpha\left(2\sqrt{R\frac{E_b}{N_0}}\right)\left(1 - Q_\alpha\left(2\sqrt{R\frac{E_b}{N_0}}\right)\right)}\right)^d. \tag{8}$$

The Viterbi algorithm can be used with unquantized decisions from the detector to perform maximum likelihood decoding of the received sequence. The branch metric used for maximum likelihood decoding is the logarithm of the joint probability of the sequence conditioned on the transmitted sequence [7].

$$\mu = -\log p\left(\underline{y} \mid \underline{x}\right) = -\log \prod_t f_\alpha\left(y_t - x_t; \gamma\right)$$
$$= -\sum_t \log f_\alpha\left(y_t - x_t; \gamma\right) \tag{9}$$

where μ is the branch metric, $\underline{y}$ is the received sequence, $\underline{x}$ is the transmitted sequence for the branch and $p(\underline{y}|\underline{x})$ is the conditional probability function or conditional pdf of the received sequence conditioned on the transmitted sequence. The pair-wise probability of error is then [6]

$$p_d = P\left(\sum_{t=1}^{d} \log \frac{p\left(y_t|1\right)}{p\left(y_t|0\right)} > 0\right). \tag{10}$$

In an AWGN channel, (9) can be simplified further resulting in the Euclidean branch metric, $\mu = \sum_t (y_t - x_t)^2$, commonly used in the soft decision Viterbi algorithm. It is equivalent to the maximum likelihood metric when the channel is AWGN. However, in an AWSαSN channel, the Euclidean norm metric is not optimal. Alternative metrics such as the Huber penalty function [8] and the 1-norm metric [9], [10] have been noted for their robustness in non-Gaussian noise. The p-norm ($p < \alpha$) is often known to be a robust cost function in the presence of α-stable noise [1]. Inspired by these heuristics, we use the p-norm (with $p = 1$) branch metric $\mu = \sum_t |y_t - x_t|$ for the Viterbi algorithm in the presence of SαS noise. For a transmitted code word with weight d, we can assume $x_1 \cdots x_d$ to be 1. The probability of bit error is then given by

$$\begin{aligned} p_d &= P\left(\sum_{t=1}^{d} \{|y_t - \sqrt{RE_b}| - |y_t + \sqrt{RE_b}|\} > 0\right) \\ &= P\left(\sum_{t=1}^{d} -2g\left(y_t\right) > 0\right) \\ &= P\left(\sum_{t=1}^{d} g(\sqrt{RE_b} + n_t) < 0\right), \end{aligned}$$

where

$$g\left(y\right) = \frac{1}{2}\{|y + \sqrt{RE_b}| - |y - \sqrt{RE_b}|\}. \tag{11}$$

Although the random variable n_t has an infinite variance, the random variable $g\left(\sqrt{RE_b} + n_t\right)$ has a finite variance because $g(y)$ is bounded. For large d, the summation in (11) is the sum of a large number of finite variance independent identically distributed (i.i.d.) random variables, and hence, approximately Gaussian. The mean μ_g and variance σ_g^2 of $g\left(\sqrt{RE_b} + n_t\right)$ cannot be evaluated in closed form, but an upper bound on p_d can be found by assuming a Cauchy distribution for n_t, which underestimates the mean and overestimates the variance.

$$\begin{aligned} \mu_g &= \int_{-\sqrt{RE_b}}^{\sqrt{RE_b}} x f_g\left(x\right) dx \\ &= \sqrt{N_0}\left[\frac{2}{\pi}\sqrt{R\frac{E_b}{N_0}} \arctan\left(4\sqrt{R\frac{E_b}{N_0}}\right)\right. \\ &\qquad \left. -\frac{1}{4\pi}\log\left(1 + 16R\frac{E_b}{N_0}\right)\right] \\ \sigma_g^2 + \mu_g^2 &= \int_{-\sqrt{RE_b}}^{\sqrt{RE_b}} x^2 f_g\left(x\right) dx \\ &= N_0\left[\frac{1}{\pi}\sqrt{R\frac{E_b}{N_0}}\left(1 - \frac{1}{2}\log\left(1 + 16R\frac{E_b}{N_0}\right)\right)\right. \\ &\qquad \left. +R\frac{E_b}{N_0} - \frac{1}{4\pi}\arctan\left(4\sqrt{R\frac{E_b}{N_0}}\right)\right]. \end{aligned} \tag{12}$$

The approximate mean and variance of the summation in (11) are $d\mu_g$ and $d\sigma_g^2$ respectively. An upper bound on the bit error probability for Viterbi decoding with a 1-norm metric in an AWSαSN channel is

$$P_b < \sum_{d=d_f}^{\infty} c_d Q\left(\sqrt{d\frac{\mu_g^2}{\sigma_g^2}}\right). \tag{13}$$

Due to the Cauchy noise assumption, the previous bound is somewhat loose. To obtain a better approximation for higher values of α, we numerically computed the integrals in (12) using a numerical approximation of the pdf f_α based on [4]. Fitting a surface on the resulting numerical values of the ratio, we obtained an empirical expression for μ_g^2/σ_g^2 in terms of α and RE_b/N_0. Thus, we have an approximate bit error probability for

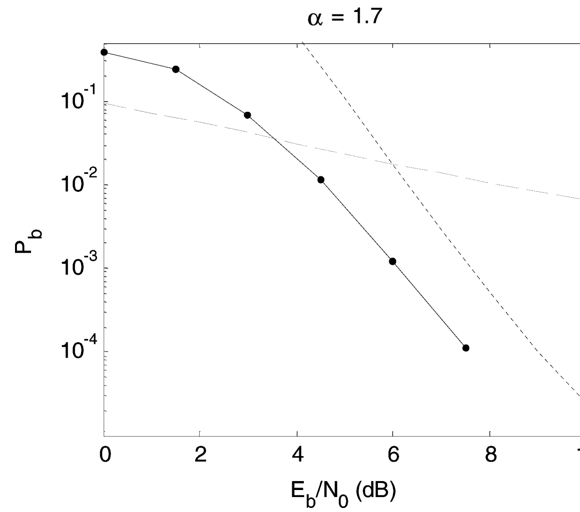

Fig. 2. Uncoded performance (dashed line), theoretical upper bound (dotted line), and simulation results (solid line with dots) for hard decision decoding.

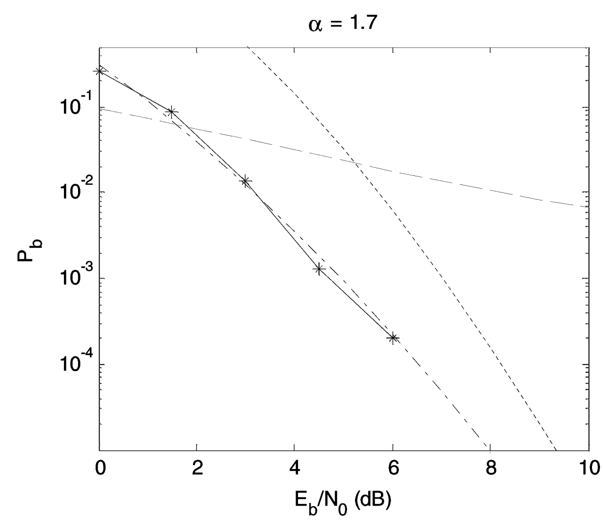

Fig. 3. Uncoded performance (dashed line), theoretical upper bound (dotted line), closed-form approximation (dash-dot line), and simulation results (solid line with dots) for 1-norm decoding.

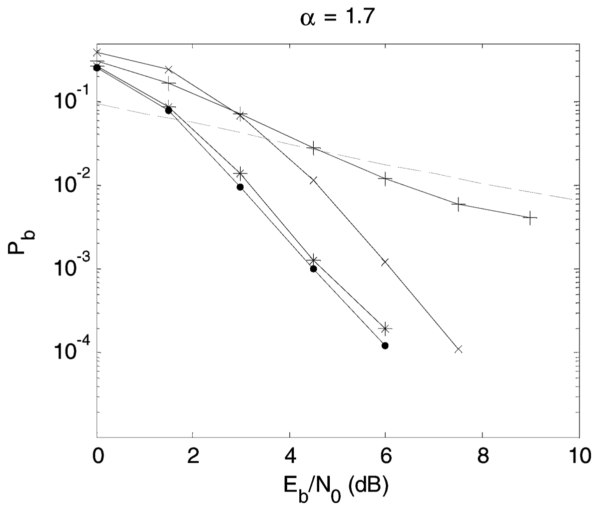

Fig. 4. Performance of uncoded system (dashed line) in comparison to coded system with hard decision decoding (solid line with cross), Euclidean metric (solid line with plus), 1-norm metric (solid line with star), and maximum likelihood decoding (solid line with dots).

Viterbi decoding with a 1-norm metric in an AWSαSN channel.

$$P_b \approx \sum_{d=d_f}^{\infty} c_d Q \times \left(\sqrt{d \exp\left(p_{21}\alpha^2 + p_{11}\alpha + p_{01}\right) \left(R\frac{E_b}{N_0}\right)^{p_{22}\alpha^2 + p_{12}\alpha + p_{02}}} \right),$$
$$p_{21} = 0.0566, \quad p_{11} = 0.3850, \quad p_{01} = -0.4077,$$
$$p_{22} = 0.0766, \quad p_{12} = 0.0417, \quad p_{02} = 0.1860. \qquad (14)$$

V. RESULTS

We simulated a coded BPSK communication system in an AWSαSN channel with a half-rate Odenwalder code. In Fig. 2, we see that the theoretical upper bound derived in (8) is approximately 1 dB higher than the simulation results for hard decision decoding. In Fig. 3, we see that the theoretical upper bound derived in (13) is approximately 2–4 dB higher than the simulation results for 1-norm decoding. The bound is loose at low E_b/N_0 and high α, and becomes tighter when the noise becomes more impulsive and at higher E_b/N_0. The closed-form approximation in (14) matches the simulation results closely. Fig. 4 compares the performance of various decoding schemes in impulsive noise. The decoding with Euclidean metric shows poor performance with little gain over the uncoded system. The hard decision decoding performs significantly better. The maximum likelihood decoding is optimal and demonstrates the best performance, approximately 2 dB better than that of the hard decision decoding. The performance of the decoding using the 1-norm metric is very close to that of the maximum likelihood decoding. As the computational complexity of the 1-norm metric is much lower than that of the maximum likelihood metric and does not require an estimate of the noise dispersion, it is a good alternative to maximum-likelihood decoding.

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