Path planning for cooperative underwater range-only navigation using a single beacon

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Abstract—Autonomous underwater vehicles (AUVs) that rely on dead reckoning suffer from unbounded localization error growth at a rate dependent on the quality (and cost) of the navigational sensors. Many AUVs surface occasionally to get a GPS position update. Alternatively underwater acoustic beacons such as long baseline (LBL) arrays are used for localization, at the cost of substantial deployment effort. The idea of cooperative localization with a few vehicles with high navigation accuracy (beacon vehicles) among a team of AUVs with poor navigational sensors has recently gained interest. Autonomous surface crafts (ASCs) with GPS, or sophisticated AUVs with expensive navigational sensors may play the role of beacon vehicles. Other AUVs are able to measure their range to these acoustically, and use the resulting information for self-localization. Since a single range measurement is insufficient for unambiguous localization, multiple beacon vehicles are usually required. In this paper, we explore the use of a single beacon vehicle to support multiple AUVs. We develop path planning algorithms for the beacon vehicle that take into account and minimize the errors being accumulated by other AUVs. We show that the generated beacon vehicle path enables the other AUVs to get sufficient information to keep their localization errors bounded over time.

Index Terms—Autonomous underwater vehicles, localization, positioning, navigation.

I. Introduction

Underwater navigation is a challenging problem that has received considerable attention in recent years [1]. As GPS signals cannot be received underwater, autonomous underwater vehicles (AUVs) often rely on sensors such as compass, Doppler Velocity Log (DVL) and Inertial Navigation System (INS) to estimate their position. Dead reckoning based on these sensors suffers from a problem of unbounded navigation error growth over time. With inexpensive sensors, this error growth is rapid, while expensive high-quality sensors result in slower (but still unbounded) error growth. To solve this problem, most AUVs today surface occasionally to get a GPS position update. Alternatively a set of underwater acoustic beacons in the form of a long baseline (LBL) array may be used for position updates at the cost of substantial deployment effort.

The idea of cooperative localization with a few vehicles that know their position well and other AUVs with poor navigational sensors is not new. The vehicles with accurate position estimates are referred to by some authors as master vehicles [2]. In other recent papers [3], [4], the authors refer to these vehicles as communication and navigation aid (CNA) vehicles. Since these vehicles serve the same purpose as radio beacons in terrestrial navigation, we simply call them *beacon vehicles*. The beacon vehicles may be autonomous surface crafts (ASCs) with access to GPS, or sophisticated AUVs with expensive high-quality navigational sensors. The other AUVs in the team are aided by the beacon vehicles; we call these the *survey AUVs* as they typically are equipped with other non-navigational sensors.

Cooperative AUVs typically need to communicate in order to cooperate. Hence they are usually fitted with underwater acoustic modems that may also be used to measure range between two vehicles using the travel time of the acoustic signals. When this range information is combined with the known positions of the beacon vehicles, the survey AUVs are able to determine their positions [4]. Since a single range measurement from a known beacon location does not allow unambiguous localization of the survey AUV, multiple beacon vehicles are usually needed. However range-only localization using a single beacon is operationally attractive, and has been explored by several researchers [2], [3], [5]–[8]. Their work includes observability analysis, algorithms for position determination based on range measurements and some experimental results. Although all of these authors acknowledge that the relative motion of the beacon vehicle and the survey AUVs is key to having single beacon range-only navigation perform well, the problem of determining the optimal path of the beacon vehicle given the desired path of the survey AUVs has received little attention. For example, the work in [2] assumes a circular path for the beacon vehicle, while [3] uses a zig-zag path during experiments. In [6] the author suggests some maneuvers for the survey AUV while stating that the beacon vehicle would "most likely sprint and drift off side the survey path to force enough relative motion change to fix vehicle position". Although [8] assumes that the path of the beacon vehicle is produced in such a way that it improves the position estimate of the survey AUV, we do not present the algorithm to do so in that paper.

In this paper we explore the possibility of having a single beacon vehicle supporting multiple survey AUVs. We develop a path planning algorithm for the beacon vehicle which takes into account and aims to minimize the localization errors being accumulated by the supported survey AUVs. We show that the path followed by the beacon vehicle using this algorithm enables the survey AUVs to get sufficient information to keep

their position error bounded over time. The work presented here is complementary and may be used with many of the range-only position estimation algorithms previously proposed by other researchers (e.g. [4], [8]).

II. PROBLEM FORMULATION

A. Optimization problem definition

We assume that the beacon vehicle knows its position accurately and transmits a beacon signal periodically, with period of τ seconds. This transmission enables all survey AUVs within acoustic range of the beacon vehicle to estimate their range from the beacon vehicle by measuring the propagation delay of the signal. Since the beacon vehicle makes a navigation decision per beacon transmission period, we represent time using an index $t \in \{0..T\}$. The elapsed time in seconds from the start of the mission to time instant t is simply $t\tau$.

Although the underwater environment is 3-dimensional, it is typical that the depth for the beacon and survey vehicles is specified in a mission and may not be altered by our path planning algorithm. We therefore represent the position of each vehicle using a 2-dimensional position vector and the direction of travel of each vehicle by a yaw angle. Let $\mathbf{x}_t^B$ be the position and ϕ_t^B be the heading of the beacon vehicle at time t. Let N be the number of survey AUVs supported by the beacon vehicle. We index the survey AUVs by $j \in \{1..N\}$. Let $\mathbf{x}_t^j$ represent the position of survey AUV j at time t. At every time index t, we have estimates $\hat{R}_t^j$ of the 2-dimensional range (easily estimated from the measured range by taking into account the difference in depths between the vehicles) between the beacon vehicle and each of the survey AUVs. We model the error in range estimation as a zero-mean Gaussian random variable with variance σ^2:

$$\hat{R}_t^j = \mathcal{N}(|\mathbf{x}_t^j - \mathbf{x}_t^B|, \sigma^2) \tag{1}$$

We further model the error in position estimation of the survey AUVs as a 2-dimensional zero-mean Gaussian random variable described by three parameters – the direction θ_t^j of minimum error, the error ϵ_t^j along direction θ_t^j, and the error $\bar{\epsilon}_t^j$ in the tangential direction. Just after a range measurement at time $t+1$, the error is minimum along the line joining the beacon and the survey vehicle:

$$\theta_{t+1}^j = \angle(\mathbf{x}_{t+1}^j - \mathbf{x}_{t+1}^B) \tag{2}$$
$$\epsilon_{t+1}^j = \sigma \tag{3}$$

The range measurement gives no information in the tangential direction and therefore the error grows in that direction. Assuming that the survey AUVs use velocity estimates for dead reckoning, the position error variance in the tangential direction will grow linearly with time:

$$(\bar{\epsilon}_{t+1}^j)^2 = \frac{(\epsilon_t^j \bar{\epsilon}_t^j)^2}{(\epsilon_t^j \cos \gamma_t^j)^2 + (\bar{\epsilon}_t^j \sin \gamma_t^j)^2} + \alpha\tau \tag{4}$$

where $\gamma_t^j = \theta_{t+1}^j - \theta_t^j$ and α is the constant of proportionality (determined by the accuracy of the velocity estimate of the survey AUV).

The navigation decision made by the beacon vehicle at each time step t is δ_t^B, the turning angle during the time interval until the next decision. If $\dot{\phi}_{\max}^B$ is the maximum turning rate,

$$|\delta_t^B| \leq \dot{\phi}_{\max}^B \tau \tag{5}$$

If s^B is the speed of the beacon vehicle then the heading and position of the vehicle at time $t+1$ is approximately given by

$$\phi_{t+1}^B = \phi_t^B + \delta_t^B \tag{6}$$
$$\mathbf{x}_{t+1}^B = \mathbf{x}_t^B + \tau s^B \begin{pmatrix} \cos \phi_{t+1}^B \\ \sin \phi_{t+1}^B \end{pmatrix} \tag{7}$$

In order to ensure that the beacon and survey vehicles do not collide but are within transmission range of each other, we require that

$$D_{\min} \leq |\mathbf{x}_{t+1}^j - \mathbf{x}_{t+1}^B| \leq D_{\max} \quad \forall j \tag{8}$$

We assume that the position of each survey AUV is known at the start of the mission with an accuracy of ϵ_0 in all directions:

$$\epsilon_0^j = \bar{\epsilon}_0^j = \epsilon_0 \tag{9}$$
$$\theta_0^j = 0 \quad \text{(arbitrary choice)} \tag{10}$$

Given the desired paths $\{\mathbf{x}_t^j \; \forall \; t\}$ of the survey AUVs and the initial position $\mathbf{x}_0^B$ and heading ϕ_0^B of the beacon vehicle, we wish to plan a path for the beacon vehicle such that we minimize the sum-square estimated position error across all survey AUVs for the entire mission duration. The path is fully determined by the sequence of decisions $\{\delta_t^B\}$ made during the mission:

$$\{\delta_t^B\} = \arg\min \sum_{j,t} \left[(\epsilon_t^j)^2 + (\bar{\epsilon}_t^j)^2 \right] \tag{11}$$

B. Dynamic programming problem representation

We can rewrite the problem definition in the previous section as a dynamic program [9]. The state S_t of the system consists of the positions and heading of the beacon vehicle and the estimated position error of each survey AUV:

$$S_t = \{\mathbf{x}_t^B, \phi_t^B, (\epsilon_t^j, \bar{\epsilon}_t^j, \theta_t^j) \; \forall \; j\} \tag{12}$$

The decision space at time t is represented by the set $\mathcal{A}(S_t)$ of turn angles δ_t^B that the beacon vehicle can adopt subject to constraints (5) and (8). Unlike typical dynamic programs, the decision space is not discrete, but continuous, and therefore the set $\mathcal{A}(S_t)$ is an infinite set. When a decision $a_t \in \mathcal{A}(S_t)$ is made, the state change is given by the state transition function $S_{t+1} = \mathcal{T}(S_t, a_t)$ defined by (2), (3), (4), (6) and (7). The decision incurs a cost $C(S_t, a_t)$ equal to the sum-square position error at time $t+1$ across all survey AUVs:

$$C(S_t, a_t) = \sum_j \left[(\epsilon_{t+1}^j)^2 + (\bar{\epsilon}_{t+1}^j)^2 \right] \tag{13}$$

We wish to find an optimal policy (representing a route for the beacon vehicle, comprising of a sequence of decisions $\{a_t \in$

$\mathcal{A}(S_t) \; \forall \; t \in \{0..T-1\}\})$ that will minimize the total cost over the mission duration:

$$\{a_t\} = \arg\min_{\{a_t\}} \sum_{t=0}^{T-1} C(S_t, a_t) \qquad (14)$$

A dynamic programming problem naturally lends itself to a recursive formulation based on Bellman's principle of optimality [9]. The Bellman's equation introduces a value function $V_t(S_t)$ that represents the cost of applying an optimal policy from a given state S_t until the end of the mission:

$$\begin{aligned}
V_t(S_t) &= \min_{\{a_{t'}\}} \sum_{t'=t}^{T-1} C(S_{t'}, a_{t'}) \qquad (15) \\
&= \min_{a \in \mathcal{A}(S_t)} \left[C(S_t, a) + \min_{\{a_{t'}\}} \sum_{t'=t+1}^{T-1} C(S_{t'}, a_{t'}) \right] \\
&= \min_{a \in \mathcal{A}(S_t)} \left[C(S_t, a) + V_{t+1}(S_{t+1}) \right]
\end{aligned}$$

where $S_{t+1} = \mathcal{T}(S_t, a)$. The optimal decision at any point in the mission is then simply given by:

$$a_t = \arg\min_{a \in \mathcal{A}(S_t)} \left[C(S_t, a) + V_{t+1}(S_{t+1}) \right] \qquad (16)$$

III. DECISION SPACE APPROXIMATION

Problems of the form described in the previous section are typically solved using standard methods in dynamic programming such as value iteration or policy iteration [9]. Unfortunately, these methods cannot be applied to our problem as our state space and our decision space are continuous (and the corresponding sets of states and decisions are each infinite). We therefore resort to approximate methods of solving this problem.

The problem of a continuous decision space can be solved by discretizing the decision space yielding a finite set of decisions that can be made at each stage. The small dimensionality and the constraints placed on the decisions allow us to approximate the decision space by a set of discrete decisions distributed in the space. The decision space is then approximated by the finite set $\tilde{\mathcal{A}}(S_t)$ consisting of A discrete turn angles uniformly spaced in the range $-\dot{\phi}_{\max}^B \tau$ to $\dot{\phi}_{\max}^B \tau$ and satisfying constraint (8).

Implementation note: For some states S, the decision set $\tilde{\mathcal{A}}(S)$ may be an empty set ($\emptyset$). This happens when all potential decisions fail to satisfy constraint (8). In order to allow the algorithm to recover from such states, if $\tilde{\mathcal{A}}(S_t) = \emptyset$, we allow $\tilde{\mathcal{A}}(S)$ to contain a single decision that violates the constraint by the least amount. This ensures that the algorithm can continue and potentially recover to a state S' where $\tilde{\mathcal{A}}(S') \neq \emptyset$.

IV. VALUE FUNCTION APPROXIMATIONS

The approach of discretizing the space, unfortunately, cannot be used for the state space. The state space has a high dimensionality and some of the dimensions represent a physically large geographical area. Discretizing it to a reasonable approximation would require a very large set of discrete states, leading to computationally unfeasible solutions.

Perhaps we can evaluate $V_t(S_t)$ on a continuous state space by using (15)? The computational load of this approach is $\mathcal{O}(A^T)$ and grows very rapidly with the length of the mission. For modest length missions and small values of A, the computational load is already prohibitively large.

We therefore seek an approximation $\tilde{V}_t(S_t)$ to the value function $V_t(S_t)$ to solve our optimization problem. The approximation can then be used to replace the value function in (16) to give:

$$a_t = \arg\min_{a \in \mathcal{A}(S_t)} \left[C(S_t, a) + \tilde{V}_{t+1}(S_{t+1}) \right] \qquad (17)$$

A. Greedy strategy

A trivial approximation for the value function is

$$\tilde{V}_t(S_t) = 0 \qquad (18)$$

Since the value function represents the future costs to be incurred from the next state to the end of the mission, setting it to zero is equivalent to ignoring these future costs and making the current decision based purely on the cost incurred by the decision. Hence we call this a *greedy strategy*. The computational load to generate an optimal route in this strategy is $\mathcal{O}(TA)$ and only increases linearly with the length of the mission.

B. L-level look-ahead strategy

A better approximation for the value function $V_t(S_t)$ is given by evaluating only L terms of the summation in (15):

$$\tilde{V}_t(S_t) = \min_{\{a_{t'}\}} \sum_{t'=t}^{t+L-1} C(S_{t'}, a_{t'}), \quad L \in \mathbb{Z}^+ \qquad (19)$$

The computational load to generate an optimal route in this strategy is $\mathcal{O}(TA^{L+1})$ and can be traded-off against accuracy by varying the look-ahead level L. This computational load only increases linearly with the length of the mission, but can be significantly higher than that of the greedy strategy (which can be viewed as a 0-level look-ahead strategy). For brevity, we denote a L-level look ahead strategy as LA-L.

V. SIMULATION RESULTS

To get an idea of the expected localization performance of survey AUVs supported by a single beacon vehicle, we tested the greedy strategy and the L-level look-ahead strategy through numerical simulations. The simulations were based on the parameters shown in Table I.

A. Single survey AUV positioning

A survey AUV was given a mission to execute a lawn-mover pattern survey at a cruising speed of 1.5 m/s with each leg of the survey being 500 m as shown in Fig. 1. The beacon vehicle was deployed about 140 m away and allowed to cruise at the same speed as the survey AUV (s^B = 1.5 m/s). Its path was determined through the various strategies described above. For each of the paths generated by the strategies, the positioning

TABLE I
DEFAULT PARAMETERS FOR SIMULATIONS

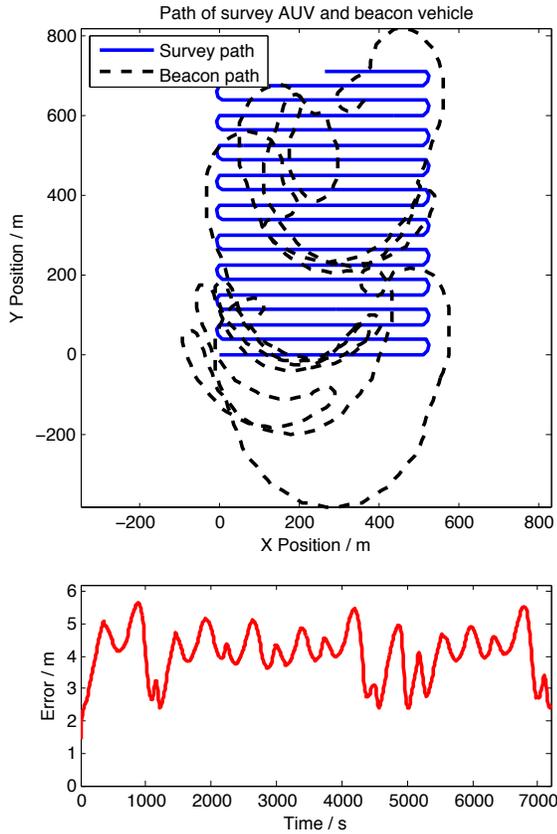| Parameter	Value
τ	10 s
T	720 (2 hours)
σ	1 m
$\dot{\phi}_{max}^{B}$	0.07 rad/s
D_{min}	100 m
D_{max}	1000 m
ϵ_0	1 m
α	0.1 m^2/s

Fig. 1. Path of survey AUV, beacon vehicle and estimated positioning error over time. The beacon vehicle path was generated for $A = 5$, greedy strategy and $s^B = 1.5$ m/s

error of the survey AUV was estimated over the entire mission duration. The path generated by the greedy algorithm for $A = 5$ is shown in Fig. 1, along with the estimated error over time. Without a supporting beacon vehicle, we expect that error in position of the survey AUV to grow without bound with time to about 27 m in 2 hours. However, with a single beacon vehicle, this error remains bounded.

The performance for the different strategies and varying values of A is summarized in Table II. We can see that performance generally improves with increased A and increased look-ahead level L. Increasing L to beyond 4 does not seem to improve performance significantly, while it increases the computational load tremendously. At LA-2 and above,

TABLE II
SIMULATION RESULTS FOR 1 AUV SURVEY MISSION ($s^B = 1.5$ M/S)

Strategy	A	Error (m)		Distance (m)		
		RMS	Max	Min	Mean	Max
Greedy	3	4.8	5.9	107.8	551.5	899.1
LA-1	3	4.4	5.3	109.3	449.8	655.4
LA-2	3	3.6	4.9	101.9	242.5	480.8
LA-3	3	3.6	5.0	105.0	235.7	502.9
LA-4	3	3.6	4.6	100.0	234.3	417.9
LA-5	3	3.5	4.5	100.1	184.4	287.2
Greedy	5	4.2	5.7	98.8	352.9	615.0
LA-1	5	3.9	5.2	100.2	309.8	596.7
LA-2	5	3.5	4.7	100.2	207.8	417.9
LA-3	5	3.6	4.6	100.6	237.8	417.1
LA-4	5	3.5	4.6	100.1	210.5	415.3
Greedy	7	4.1	5.7	101.4	347.6	594.9
LA-1	7	3.9	5.2	100.2	290.6	599.4
LA-2	7	3.5	4.7	100.0	206.6	416.1
LA-3	7	3.6	4.6	100.4	236.8	421.7

increasing A offers very small performance improvement. The best performance-to-computation trade-off is offered by $A = 3$ and LA-4. With these parameters, the error remains bounded below 4.6 m with a root-mean-square (RMS) error of 3.6 m.

It is interesting to note that the mean distance between the survey AUV and the beacon vehicle generally reduces with increasing look-ahead level. The beacon vehicle effectively "learns" that it is able to control the error of the survey AUV better by being closer, since it can effect a larger change in relative bearing between the vehicles by traveling the same distance at a shorter range. It is then natural to ask if the positioning accuracy would improve if the beacon vehicle was allowed to move faster. In Table III, we present results from simulations where the beacon vehicle moves faster ($s^B = 2$ m/s) than the survey AUV. We indeed see that the performance is somewhat improved, especially at higher look-ahead levels, with the RMS error in case of $A = 3$ and LA-4 reducing from 3.6 m to 3.2 m. The resulting path generated is shown in Fig. 2.

TABLE III
SIMULATION RESULTS FOR 1 AUV SURVEY MISSION ($s^B = 2$ M/S)

Strategy	A	Error (m)		Distance (m)		
		RMS	Max	Min	Mean	Max
LA-3	3	3.3	4.6	100.9	193.2	500.7
LA-4	3	3.2	4.2	100.1	172.6	301.9
LA-5	3	3.1	3.9	100.3	144.5	228.9
LA-2	5	3.5	4.8	100.0	251.8	548.1
LA-3	5	3.2	4.6	100.1	182.9	444.5
LA-4	5	3.1	4.1	100.1	147.0	273.2

B. Multiple survey AUVs positioning

We next turn our attention to the simulation scenario where a single beacon vehicle supports multiple survey AUVs. Two survey AUVs were assigned missions to execute lawn-mover pattern surveys at cruising speeds of 1.5 m/s. The AUVs were assigned adjacent survey areas with each leg of the survey
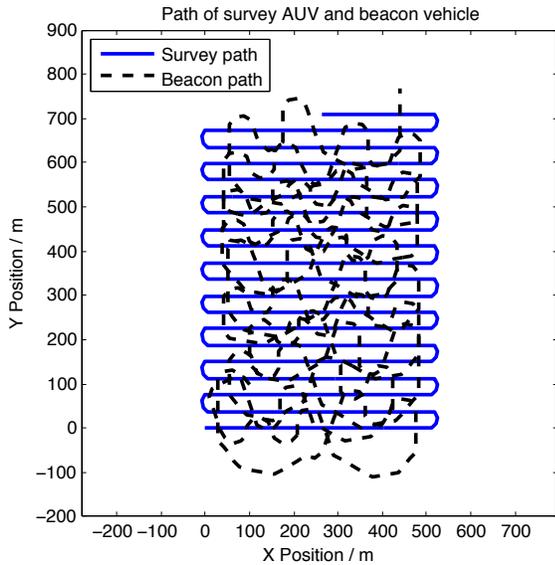
Fig. 2. Path of survey AUV and beacon vehicle for $A = 3$, LA-4 strategy and $s^B = 2$ m/s

being 300 m as shown in Fig. 3. The beacon vehicle was deployed at a point between the two survey AUVs.

Results from several simulations of this scenario with various parameters are shown in Table IV. When the beacon vehicle is supporting multiple survey AUVs, not surprisingly, we find that the RMS error for each survey AUV is higher than the case with a single supported survey AUV. However, the error is still bounded and relatively small (about 4-5 m for the LA-4 and $A = 3$). The performance of the L-5 algorithms is slightly better than L-4. However, at L-5, an increase in A only provides marginal benefit at huge computational cost. The best performing path of the beacon vehicle (L-5 and $A = 5$) is shown in Fig. 3.
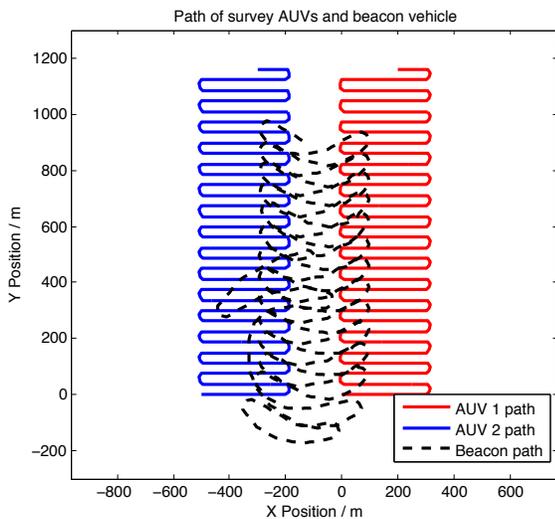

Fig. 3. Path of 2 survey AUVs and beacon vehicle for $A = 5$, LA-5 strategy and $s^B = 2$ m/s

Strategy	A	s^B (m/s)	RMS Error (m) AUV1	RMS Error (m) AUV2	Max Error (m) AUV1	Max Error (m) AUV2
LA-4	3	1.5	5.3	4.9	6.1	5.9
LA-4	3	2.0	4.1	4.0	5.0	4.7
LA-4	5	2.0	4.1	3.9	5.1	4.8
LA-4	7	2.0	4.1	3.9	5.0	4.8
LA-5	3	2.0	3.9	4.0	5.1	5.0
LA-5	5	2.0	4.0	3.8	4.9	4.7

VI. ONLINE V/S OFFLINE MODE

The algorithm presented here can be used in an *offline* mode or an *online* mode. In the offline mode, the planned survey AUV paths are fed to the algorithm to generate a path for the beacon vehicle. This beacon vehicle then follows this path during the mission. Every τ seconds, the beacon vehicle transmits a beacon signal with its position and time-stamp, and all survey AUVs within acoustic range estimate their range to the beacon vehicle based on the propagation delay of this signal. If accurate time synchronization is not available, two-way propagation delay measurements may be required to estimate range for each survey AUV. The survey AUVs then use the range information and the received position of the beacon vehicle as inputs to an appropriate position estimation algorithm such as an Extended Kalman Filter (EKF) to update the estimate of their position [8]. In case of limited underwater communication capability, the planned path of the beacon vehicle can be made known to all the survey AUVs before the start of the mission, and the position of the beacon vehicle at any time can be estimated rather than transmitted acoustically.

The offline mode suffers from the drawback that the actual mission may deviate from the planned mission due to uncertainties that are omnipresent in the marine environment. An online version of the algorithm can help remove this drawback. In the online version, the beacon vehicle makes navigation decisions using this algorithm during the mission. The survey AUV paths can be made known to the beacon vehicle in advance, and updates can be transmitted during the mission if required. As in the offline mode, the beacon vehicle transmits the beacon signal with its position every τ seconds. This signal is used by the survey AUVs to update their own position estimates. If the survey AUVs have an estimate of their position errors, these estimates can be transmitted back to the beacon vehicle and used to update the state information in the algorithm. The online mode is capable of responding to deviations from plan and other uncertainties during the mission, at the cost of more communication and in-vehicle computational capability.

VII. DISCUSSION AND FUTURE WORK

In this paper, we presented an algorithm for path planning for a beacon vehicle that can support one or more survey AUVs in underwater localization via range measurements. The algorithm aims to minimize the accumulated position error for

all AUVs that it is supporting. Through simulation studies we have shown that the algorithm ensures that each of the supported AUVs receives range information from time-varying locations such that the position errors can be bounded. With appropriate choice of parameters, the algorithm can have a low computational load and yet produce paths with good performance.

The model assumes that the beacon vehicle knows its position accurately. It also assumes a simple error model for the range measurement and dead reckoning, and an approximate motion model for the beacon vehicle. In practice some of these assumptions may not be valid, and deviations may lead to larger position errors than those suggested by the simulation studies. We are currently in the process of testing the algorithms in field trials with the STARFISH AUV [10], [11]. Preliminary results show that the position error indeed remains bounded during experiments using a simplified version of the greedy algorithm described here [8]. Alternate motion and error models can easily be adopted using the problem formulation presented here, if needed. Additional constraints such as geofencing have to be imposed on the path planning to ensure vehicle safety during deployments. These constraints are also easily included, but have been omitted from this paper for brevity.

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