

# Design and Performance Analysis of a Signal Detector Based on Suprathreshold Stochastic Resonance

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## Abstract

This paper presents the design and performance analysis of a detector based on suprathreshold stochastic resonance (SSR) for the detection of deterministic signals in heavy-tailed non-Gaussian noise. The detector consists of a matched filter preceded by an SSR system which acts as a preprocessor. The SSR system is composed of an array of 2-level quantizers with independent and identically distributed (i.i.d) noise added to the input of each quantizer. The standard deviation  $\sigma$  of quantizer noise is chosen to maximize the detection probability for a given false alarm probability. In the case of a weak signal, the optimum  $\sigma$  also minimizes the mean-square difference between the output of the quantizer array and the output of the nonlinear transformation of the locally optimum detector. The optimum  $\sigma$  depends only on the probability density functions (pdfs) of input noise and quantizer noise for weak signals, and also on the signal amplitude and the false alarm probability for non-weak signals. Improvement in detector performance stems primarily from quantization and to a lesser extent from the optimization of quantizer noise. For most input noise pdfs, the performance of the SSR detector is very close to that of the optimum detector.

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**Keywords** : Suprathreshold stochastic resonance, non - Gaussian noise, nonlinear detector, near - optimal detection

**Main body**

## 1. Introduction

Optimal and robust detection of signals in non-Gaussian noise is a problem of great interest in several applications such as sonar [1,2], radar [3] and watermark detection [4]. Optimal detection of a deterministic signal  $As(t)$  in additive white noise  $i(t)$  involves computation of the following test statistic [5]

$$T_{od}(x(t)) = \sum_{t=0}^{N-1} \ln \left( \frac{f_i(x(t) - As(t))}{f_i(x(t))} \right), \quad (1)$$

where  $\mathbf{x} = [x(0) \dots x(N-1)]^T$  is the data vector and  $f_i(\cdot)$  is the probability density function (pdf) of noise. The test statistic of the locally optimum (LO) detector for the detection of a weak signal ( $A \rightarrow 0$ ) is generated by the nonlinear correlator

<sup>1</sup> Abbreviations: SD: Standard deviation, NQ: Noisy Quantizer, BQ: Binary Quantizer, GM: Gaussian mixture, GG: Generalized Gaussian, CGM: Cauchy-Gaussian mixture

$$T_{LO}(\mathbf{x}) = \sum_{t=0}^{N-1} L_i(x(t))s(t), \quad (2)$$

where

$$L_i(x) = -(df_i(x)/dx) / f_i(x) \quad (3)$$

is a memoryless transformation which may be called the locally optimal (LO) transform. Both these detectors reduce to a simple replica correlator or matched filter if the noise is Gaussian. But if the noise is non-Gaussian, the transformations defined in (1) and (2) are nonlinear functions and their implementation becomes a computationally difficult task. These detectors also require the prior knowledge of the noise pdf, and hence their performance is sensitive to errors in modeling the noise pdf. It is therefore of interest to design near-optimal detectors which are easy to implement and robust with respect to noise modeling errors.

A strategy that has been widely used for the design of simple suboptimal detectors in non-Gaussian noise is based on optimal quantization to approximate the nonlinear transformation associated with the optimal detector [6-9]. This approach is prompted by the fact that the LO transform  $L_i(x)$  reduces to a simple one-bit quantization if the noise is Laplacian. Another approach that has received much attention in recent years involves a constructive use of noise to aid detection [10-24]. This approach is based on the phenomenon of stochastic resonance (SR) [25], which may be defined as a non-monotonic variation of a system performance measure with respect to input noise intensity. The performance measure that is of interest to us is probability of detection, although other performance measures such as output SNR [25,26], SNR gain [14,16], mutual (Shannon) information [27], Kullback entropy [28], Fisher information [29], and cross-correlation [30] have also been considered in other applications of SR. SR is exhibited by dynamic bistable systems and also by static (memoryless) systems with threshold nonlinearity [26]. A two-level quantizer is the simplest example of the latter class of systems. SR may be realized either by adding noise or by tuning the quantizer threshold [31]. But the realization of SR is limited to situations wherein the signal alone is too weak either to trigger a transition from one state to another or to advertise its presence in some other fashion. Stocks [32] showed that the weak signal constraint can be overcome if a single quantizer is replaced by an array of quantizers and the primary input is supplemented by an additional independent noise at each quantizer. The additional noises at the quantizer inputs are independent and identically distributed (i.i.d), and the output of the array is obtained by averaging the outputs from all quantizers. This extended version of SR is called suprathreshold stochastic resonance (SSR). Detectors based on a noisy quantizer array retain the inherent simplicity of quantizer-based detectors and achieve better performance by utilizing the diversity provided by the injected noises.

The use of SSR for detection of deterministic signals in non-Gaussian noise was first proposed by Rousseau et al [18]. They proposed a simple test statistic obtained by correlating a replica of the signal with the output of the SSR system. Chen et al [20] have derived sufficient conditions for improvability of performance of a fixed detector by adding noise. Chen and Varshney [21] have considered the SR noise optimization problem for detectors with variable quantizer threshold. Patel and Kosko [22] have derived necessary and sufficient conditions for the existence of NP-optimal SR noise. In a subsequent paper [23], they have also determined necessary and sufficient conditions for noise-enhanced detection of deterministic signals in non-Gaussian noise using quantizer arrays. Guerriero et al [24] have studied SR effect in the context of sequential detection for shift-in-mean problems. The benefits of SR also extend to noisy quantizer array-based linear mean-squared error estimation [33].

In this paper we present details of the design and performance analysis of the nonlinear correlator-detector based on SSR [18]. A preliminary version of this paper with limited treatment on the topic appeared in [19]. The motivation for this work stems from the need to avoid the computational complexity associated with the optimal and LO detectors. The SSR detector can be easily implemented using an array of noisy one-bit quantizers and a correlator. Design of the SSR detector involves optimization of the variance and the pdf of quantizer noise. The present paper is organized as follows. The mathematical model of the SSR detector is presented in Section 2. An expression for the NP-optimal variance of quantizer noise is derived in Section 3 for the case of weak signal detection. It is shown that the optimal variance depends on the pdfs of input noise and quantizer noise. It is shown in Section 4 that the quantizer noise variance that maximizes the probability of detection also minimizes the mean-square difference

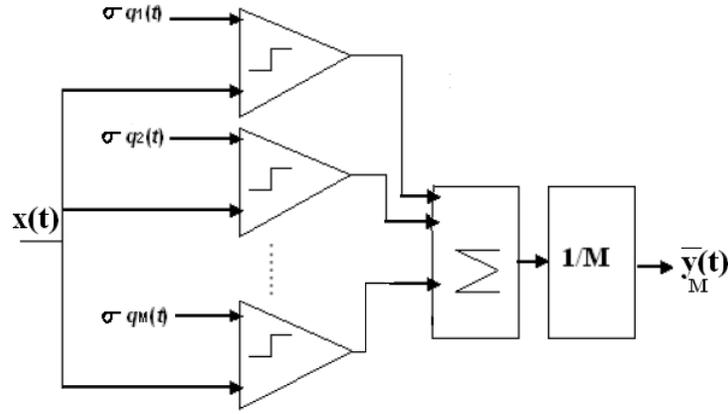


Fig. 1: Schematic diagram of SSR preprocessor

between the output of the quantizer array and the value of the LO transform defined in (3). Simulation results are presented in Section 5 for four different models of input noise pdf. The optimization issue of quantizer noise pdf is discussed in Section 6. Optimization of quantizer noise variance for detection of non-weak signals is discussed in Section 7. Conclusions are presented in Section 8.

## 2. SSR detector

Consider the detection of a signal  $As(t)$  with a known waveform  $s(t)$  and an unknown amplitude  $A > 0$ , embedded in additive white non – Gaussian noise  $i(t)$ . The data consists of  $N$  samples of  $x(t)$  and the detection problem is cast as the following hypothesis testing problem

$$H_0: x(t) = i(t)$$

$$H_1: x(t) = As(t) + i(t), t = 0, 1, \dots, N-1. \quad (4)$$

The random variables  $i(0), \dots, i(N-1)$  are i.i.d random variables with zero mean. It is assumed that the input noise  $i(t)$  has unit variance with a probability density function (pdf)  $f_i(\xi)$  and cumulative distribution function (cdf)  $F_i(\xi)$ , and that  $f_i(\xi)$  is an even function. We also assume that

$$\frac{1}{N} \sum_{t=0}^{N-1} s^2(t) = 1, \quad (5)$$

so that  $A^2$  denotes the signal power.

The SSR detector consists of an SSR preprocessor, shown in Fig. 1, followed by a replica correlator or matched filter [18,19]. The SSR preprocessor is a parallel array of  $M$  one - bit quantizers. The input to the  $m^{\text{th}}$  quantizer is  $x(t) + \sigma q_m(t)$ , where  $\{q_m(t); m = 1, 2, \dots, M\}$  are i.i.d white noise processes with unit variance, pdf  $f_q(\xi)$  which is assumed to be an even function, and cdf  $F_q(\xi)$ . The quantizer noise processes  $\{\sigma q_m(t); m = 1, 2, \dots, M\}$  are independent of  $i(t)$ . The output of the  $m^{\text{th}}$  quantizer is

$$y_m(t) = \text{sgn} [x(t) + \sigma q_m(t)], \quad (6)$$

where  $\text{sgn}(\cdot)$  denotes the signum function. The output of the SSR preprocessor is

$$\bar{y}_M(t) = (1/M) \sum_{m=1}^M y_m(t). \quad (7)$$

The preprocessor output is correlated with a replica of the signal to derive the test statistic for the SSR detector given by

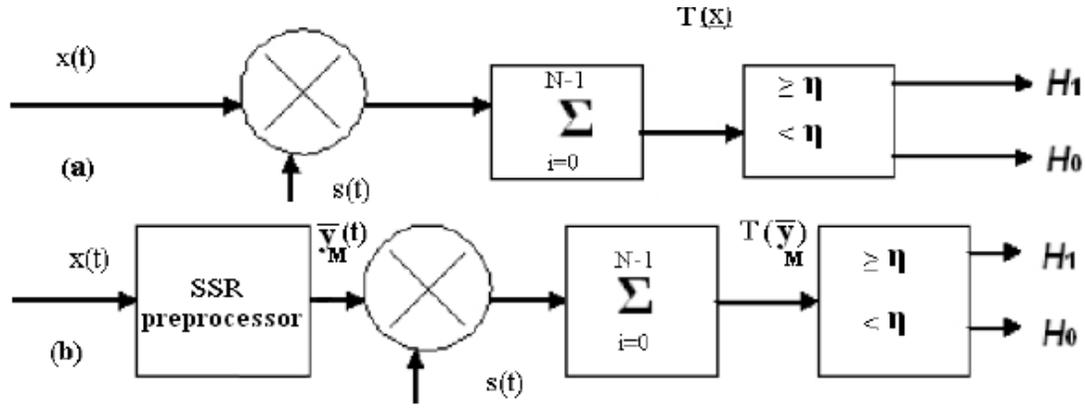


Fig. 2: (a) Linear matched filter detector. (b) Noisy quantizer (NQ) detector.  
NQ detector becomes SSR detector if  $\sigma = \sigma_{opt}$

$$T_{SSR}(\mathbf{x}) = T(\bar{\mathbf{y}}_M) = \sum_{t=0}^{N-1} \bar{y}_M(t) s(t), \quad (8)$$

where  $\bar{\mathbf{y}}_M = [\bar{y}_M(0), \dots, \bar{y}_M(N-1)]^T$  is the preprocessor output vector, and  $\mathbf{x} = [x(0), \dots, x(N-1)]^T$  is the data vector. Thus the test statistic of the SSR detector has the same form as that of the conventional linear matched filter:

$$T(\mathbf{x}) = \sum_{t=0}^{N-1} x(t) s(t). \quad (9)$$

However, the SSR detector is a nonlinear detector since  $\bar{y}_M(t)$  is a nonlinear function of  $x(t)$ . The schematic diagrams of the linear matched filter and the SSR detector are shown in Figs. 2(a) and 2(b) respectively. Since the test statistic  $T_{SSR}(\mathbf{x})$  is a nonlinear function of  $\mathbf{x}$ , it is not possible to derive analytical expressions for the pdf of  $T_{SSR}(\mathbf{x})$  under the hypotheses  $H_0$  and  $H_1$ . However, it is possible to do an asymptotic ( $N \rightarrow \infty$ ) performance analysis of the SSR detector by invoking the classical central limit theorem. Let

$$m_j = E[ T_{SSR}(\mathbf{x}); H_j ], \quad (10)$$

and

$$\lambda_j^2 = V[ T_{SSR}(\mathbf{x}); H_j ], \quad (11)$$

denote, respectively, the means and variances of  $T_{SSR}(\mathbf{x})$  under the hypotheses  $H_j$  ( $j = 0, 1$ ). We then have the following asymptotic expressions for the probability of false alarm  $P_F$  and probability of detection  $P_D$ :

$$P_F = \Phi((\eta - m_0)/\lambda_0), \quad (12)$$

$$P_D = \Phi((\eta - m_1)/\lambda_1) = \Phi((\lambda_0/\lambda_1)\Phi^{-1}(P_F) - d), \quad (13)$$

where  $\eta$  is the detector threshold,  $\Phi(\cdot)$  is the complementary cdf of a standard normal distribution,  $\Phi^{-1}(\cdot)$  denotes its inverse, and the quantity  $d$ , called deflection coefficient, is defined as

$$d = (m_1 - m_0)/\lambda_1. \quad (14)$$

The quantities  $m_j$  and  $\lambda_j$  ( $j = 0, 1$ ) depend on the pdfs  $f_0(\xi)$  and  $f_1(\xi)$  and also on the standard deviation (SD)  $\sigma$  of the quantizer noise. Under the assumption that the signal is weak ( $0 < A \ll 1$ ), it is shown in Section III that  $\lambda_0 = \lambda_1$  and  $d = \sqrt{e} G_{iq}(\sigma)$  where  $e$  is the signal energy and the quantity  $G_{iq}(\sigma)$  is defined in (29). Hence,  $P_D$  is maximized for a given

$P_F$  if  $\sigma$  is chosen to maximize  $G_{iq}(\sigma)$ . The detector in Fig. 2(b) will be designated as the noisy quantizer (NQ) detector for arbitrary  $\sigma$ . The NQ detector that uses the optimal value of  $\sigma$  (which maximizes  $G_{iq}(\sigma)$ ) is designated as the SSR detector.

### 3. Optimization of quantizer noise variance for weak signal detection

It can be shown, by using (6) and (7) and invoking the assumption that  $\{q_m(t); m = 1, 2, \dots, M\}$  are i.i.d white noises independent of the white noise  $i(t)$ , that [18,19]

$$E[\bar{y}_M(t); H_j] = 1 - 2 \int_{-\infty}^{\infty} \{F_q(-\xi/\sigma)\} f_i(\xi - A_j s(t)) d\xi, \quad (15)$$

$$E[\bar{y}_M^2(t); H_j] = 1 + 4(1-1/M) \int_{-\infty}^{\infty} F_q^2(-\xi/\sigma) - F_q(-\xi/\sigma) f_i(\xi - A_j s(t)) d\xi, j=0,1 \quad (16)$$

where  $A_0 = 0, A_1 = A$ . If the signal is weak, i.e,  $0 < A \ll 1$ , we can use the approximation

$$f_i(\xi - A_j s(t)) = f_i(\xi) - A_j s(t) [df_i(\xi)/d\xi]. \quad (17)$$

On substituting (17) into (15) and (16) and using the symmetry assumptions  $f_i(\xi) = f_i(-\xi)$  and  $f_q(\xi) = f_q(-\xi)$ , we obtain

$$E[\bar{y}_M(t); H_j] = A_j s(t) K_{iq}(\sigma), \quad (18)$$

$$V(\bar{y}_M(t); H_j) = E[\bar{y}_M^2(t)] - \{E[\bar{y}_M(t)]\}^2 = 1 - (1-1/M) U_{iq}(\sigma), \quad (19)$$

where

$$K_{iq}(\sigma) = 4 \int_0^{\infty} f_q(\xi) f_i(\sigma\xi) d\xi, \quad (20)$$

$$U_{iq}(\sigma) = 8\sigma \int_0^{\infty} F_q(\xi) \{1 - F_q(\xi)\} f_i(\sigma\xi) d\xi. \quad (21)$$

On combining (20) and (21) with (8), (10) and (11), we obtain

$$m_0 = E[T_{SSR}(\mathbf{x}); H_0] = 0, \quad (22)$$

$$m_1 = E[T_{SSR}(\mathbf{x}); H_1] = ANK_{iq}(\sigma), \quad (23)$$

$$\lambda_0^2 = V(T_{SSR}(\mathbf{x}); H_0) = N[1 - (1-1/M)U_{iq}(\sigma)], \quad (24)$$

$$\lambda_1^2 = V(T_{SSR}(\mathbf{x}); H_1) = \lambda_0^2, \quad (25)$$

On substituting (22) – (25) into (14) we obtain

$$d = \sqrt{e} G_{iq,M}(\sigma), \quad (26)$$

where

$$e = NA^2, \quad (27)$$

$$G_{iq,M}(\sigma) = K_{iq}(\sigma) / \sqrt{1 - (1 - 1/M)U_{iq}(\sigma)}. \quad (28)$$

It is seen that, as  $M$  is increased  $G_{iq,M}(\sigma)$  increases monotonically, and asymptotically attains the value

$$G_{iq}(\sigma) = K_{iq}(\sigma) / \sqrt{1 - U_{iq}(\sigma)}. \quad (29)$$

One can make the approximations  $G_{iq,M}(\sigma) = G_{iq}(\sigma)$  and  $d = \sqrt{e} G_{iq}(\sigma)$ , if  $M$  is sufficiently large. Hence, substituting (25), (26) and (28) into (13) and approximating  $G_{iq,M}(\sigma)$  by  $G_{iq}(\sigma)$ , we find that  $P_D$  is maximized for a given  $P_F$ ,  $e$ ,  $f_i(\xi)$ , and  $f_q(\xi)$  by choosing  $\sigma$  so as to maximize  $G_{iq}(\sigma)$ . Define

$$\sigma_{opt} = \arg \max G_{iq}(\sigma), \quad (30)$$

$$G_{iq,SSR} = G_{iq}(\sigma_{opt}). \quad (31)$$

For input noise of SD = 1, we have deflection coefficient  $d = \sqrt{e}$  for the matched filter,  $d = \sqrt{e} G_{iq}(\sigma)$  for the NQ detector, and  $d = \sqrt{e} G_{iq,SSR}$  for the SSR detector. Hence,  $G_{iq}(\sigma)$  and  $G_{iq,SSR}$  may be considered as the processing gain of the NQ detector and the SSR preprocessor respectively. Usually,  $G_{iq,SSR} > 1$  if the input noise is leptokurtic (kurtosis  $> 0$ ), thereby indicating that the SSR detector performs better than the matched filter in leptokurtic noise. A noisy quantizer array with  $\sigma \neq \sigma_{opt}$  provides a lower processing gain  $G_{iq}(\sigma) < G_{iq,SSR}$ . We also note that  $G_{iq}(\sigma)$  is the ideal processing gain for  $M \rightarrow \infty$ , and that the realizable processing gain for finite  $M$  is  $G_{iq,M}(\sigma)$  defined in (28).

When  $M = 1$ ,  $G_{iq,M}(\sigma)$  reduces to  $K_{iq}(\sigma)$ , which is a decreasing function of  $\sigma$  with the maximum at  $\sigma = 0$ . This result implies that, for a single quantizer, the SSR system shows no improvement with addition of quantizer noise, and that more than one quantizer is needed for the detector to benefit from addition of noise. This result has also been derived by Patel and Kosko [23] using a different approach.

#### 4. Relation between SSR detector and locally optimal detector

In this section we shall consider the relationship between the SSR detector and the LO detector [3] for the detection of a weak signal in non – Gaussian noise. The test statistic of the SSR detector is defined in (8). Since  $\{q_m(t); m = 1, 2, \dots, M\}$  are i.i.d random variables for each  $t$ , the sequence  $\{\bar{y}_M(t); m = 1, 2, \dots, M\}$  converges in probability to the conditional mean of the quantizer output:

$$\begin{aligned} \lim_{M \rightarrow \infty} \bar{y}_M(t) &\xrightarrow{P} E[\text{sgn}(x + \sigma q_m(t)) | x(t)] \\ &= 1 - 2 F_q(-x(t)/\sigma) = 2 F_q(x(t)/\sigma) - 1. \end{aligned} \quad (32)$$

The last equality in (32) follows from the assumption that  $f_q(\xi)$  is an even function. We refer to the transformation defined in (32) as the noisy quantization (NQ) transform and denote it by  $Q_q(x(t); \sigma)$ :

$$Q_q(x; \sigma) = 2 F_q(x/\sigma) - 1. \quad (33)$$

The LO transform (3) depends on the pdf  $f_i(\xi)$  of input noise while the NQ transform (33) depends on the pdf  $f_q(\xi)$  and the SD  $\sigma$  of quantizer noise. The output of the SSR system converges in probability to the NQ transform as the number of quantizers  $M$  tends to infinity. The mean-square difference (MSD) between the normalized LO transform and the normalized NQ transform under hypothesis  $H_j(j = 0,1)$  is given by

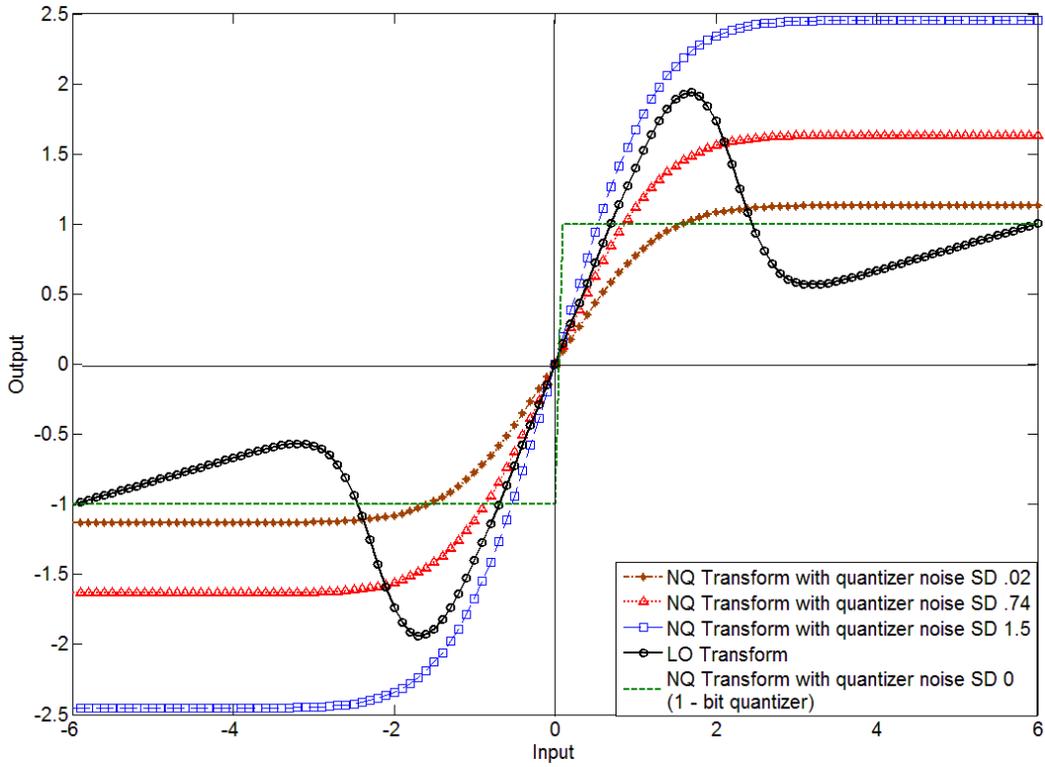


Fig. 3: Nonlinear characteristics of NQ transform with different non optimal values of  $\sigma$ , SR transform and LO transform. Input noise: GM ( $u = 0.1$ ), quantizer noise: Gaussian

$$J_{iq}(\sigma; H_j) = E \left[ \left\{ \frac{Q_q(x(t); \sigma)}{\sqrt{E[Q_q^2(x(t); \sigma); H_j]}} - \frac{L_i(x(t))}{\sqrt{E[L_i^2(x(t)); H_j]}} \right\}^2 ; H_j \right] = 2 - 2 \left[ \int_{-\infty}^{\infty} Q_q^2(\xi; \sigma) f_i(\xi - A_j s(t)) d\xi \right]^{-1/2} \times \{E[L_i^2(x(t)); H_j]\}^{-1/2} \int_{-\infty}^{\infty} Q_q(\xi; \sigma) L_i(\xi) f_i(\xi - A_j s(t)) d\xi. \quad (34)$$

In (34),  $E[L_i^2(x(t)); H_j]$  is independent of  $\sigma$ . The remaining terms in (34) are independent of the hypothesis if the signal is weak. Hence, under the weak signal approximation,  $J_{iq}(\sigma; H_j)$  is independent of  $H_j$  and minimizing  $J_{iq}(\sigma; H_j)$  is equivalent to maximizing

$$\tilde{G}_{iq} \sigma \int_{-\infty}^{\infty} Q_q(\xi; \sigma) L_i(\xi) f_i(\xi) d\xi / \sqrt{\int_{-\infty}^{\infty} Q_q^2(\xi; \sigma) f_i(\xi) d\xi}. \quad (35)$$

On substituting (3) and (33) into (35) we obtain

$$\tilde{G}_{iq} \sigma \quad K_{iq}(\sigma) / \sqrt{1 - U_{iq}(\sigma)} = G_{iq}(\sigma), \quad (36)$$

where  $G_{iq}(\sigma)$  is the processing gain defined in (29). Thus, the optimum value of  $\sigma$  under both the hypotheses, given by

$$\sigma_{opt} = \arg \min J_{iq}(\sigma) = \arg \max G_{iq}(\sigma), \quad (37)$$

simultaneously maximizes the processing gain  $G_{iq}(\sigma)$  and minimizes mean-square difference  $J_{iq}(\sigma)$ . We shall designate the NQ transform of  $x(t)$  corresponding to the optimum value of  $\sigma$  as the SR transform, and denote it by

$$S_{iq}(x(t)): S_{iq}(x) = Q_q(x; \sigma_{opt}). \quad (38)$$

The SR transform depends on the pdfs of input noise as well as the quantizer noise. For given  $f_i(\xi)$  and  $f_q(\xi)$ , the SR transform represents the closest that an SSR system consisting of a parallel array of one – bit quantizers can approach an LO transform.

The input – output characteristics of the normalized LO transform are compared with those of the normalized NQ transform for different values of  $\sigma$  in Fig. 3. The input noise  $i(t)$  has the Gaussian mixture (GM) pdf (defined in (39)) with  $u = 0.1$ , and the quantizer noise  $q(t)$  is Gaussian. For  $\sigma = 0$ , the NQ transform reduces to one – bit quantization. The mean-square difference  $J_{iq}(\sigma)$  between the LO transform and the NQ transform attains the minimum value of  $J_{iq-min}$  for the value of  $\sigma_{opt} = 0.74$ , and this optimum NQ transform is the SR transform defined in (38). It may be noted from Fig. 3 that the difference between the outputs of LO transform and SR transform is small for small values of the input where most of the probability mass of the input noise pdf  $f_i(\xi)$  is concentrated. For a given input noise distribution,  $J_{iq-min}$  varies when the pdf  $f_q(\xi)$  is varied and  $J_{iq-min}$  may be minimized through an optimal choice of  $f_q(\xi)$ . This issue is discussed in greater detail in Section VI.

It follows from the foregoing analysis that the SSR detector may be considered as a two – stage approximation to the LO detector. The LO transform is approximated by the SR transform, and the SR transform is asymptotically approached by the SSR system as  $M \rightarrow \infty$ . This interpretation suggests the possibility of further improvement of the performance of an SSR detector by other methods, such as using a higher number of quantization levels, using quantizers with different threshold levels, or using an array of any other low - complexity SR devices (if available) in order to achieve a closer fit between the SR transform and the LO transform.

## 5. Results and Discussion

### 5.1 Input noise models

Theoretical and simulation results have been obtained for four different heavy-tailed input noise distributions, viz. Gaussian mixture (GM), generalized Gaussian (GG), Cauchy-Gaussian mixture (CGM) and the Students t-distribution (ST). We consider the following single-parameter family of two-component GM pdfs of a random variable  $\xi$  with unit variance

$$f_{GM}(\xi, u) = u f_{Gauss}(\xi, 1/2u) + (1-u) f_{Gauss}(\xi, 1/2(1-u)), \quad 0 < u \leq 0.5. \quad (39)$$

where  $f_{Gauss}(\xi, \sigma^2)$  denotes the Gaussian pdf of a zero-mean random variable  $\xi$  with variance  $\sigma^2$ , given by the expression

$$f_{Gauss}(\xi, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\xi^2}{2\sigma^2}\right). \quad (40)$$

It should be noted that  $f_{GM}(\xi, u) = f_{GM}(\xi, 1-u)$ . Hence the definition of the GM pdf has been limited to the interval  $0 < u \leq 0.5$ . The pdf maximizes the kurtosis for a given ratio of variances of the components. Kurtosis is a measure of impulsiveness. Kurtosis of a zero-mean random variable  $X$  is defined as

$$\kappa_X = E[X^4] / (E[X^2])^2 - 3. \quad (41)$$

For  $u = 0.5$ , the pdf in (39) is Gaussian with zero kurtosis, and as  $u$  is reduced, kurtosis increases monotonically. The pdf of a GG random variable of unit variance with parameter  $p$  is given by

$$f_{GG}(\xi, p) = A(p) \exp(-B(p) |\xi|^p), \quad p > 0, \quad (42)$$

$$\text{where } A(p) = p \frac{\sqrt{|(3/p)|}}{2 \left(|(1/p)|\right)^{1.5}}, \quad B(p) = p \left(\frac{|(3/p)|}{|(1/p)|}\right)^{p/2},$$

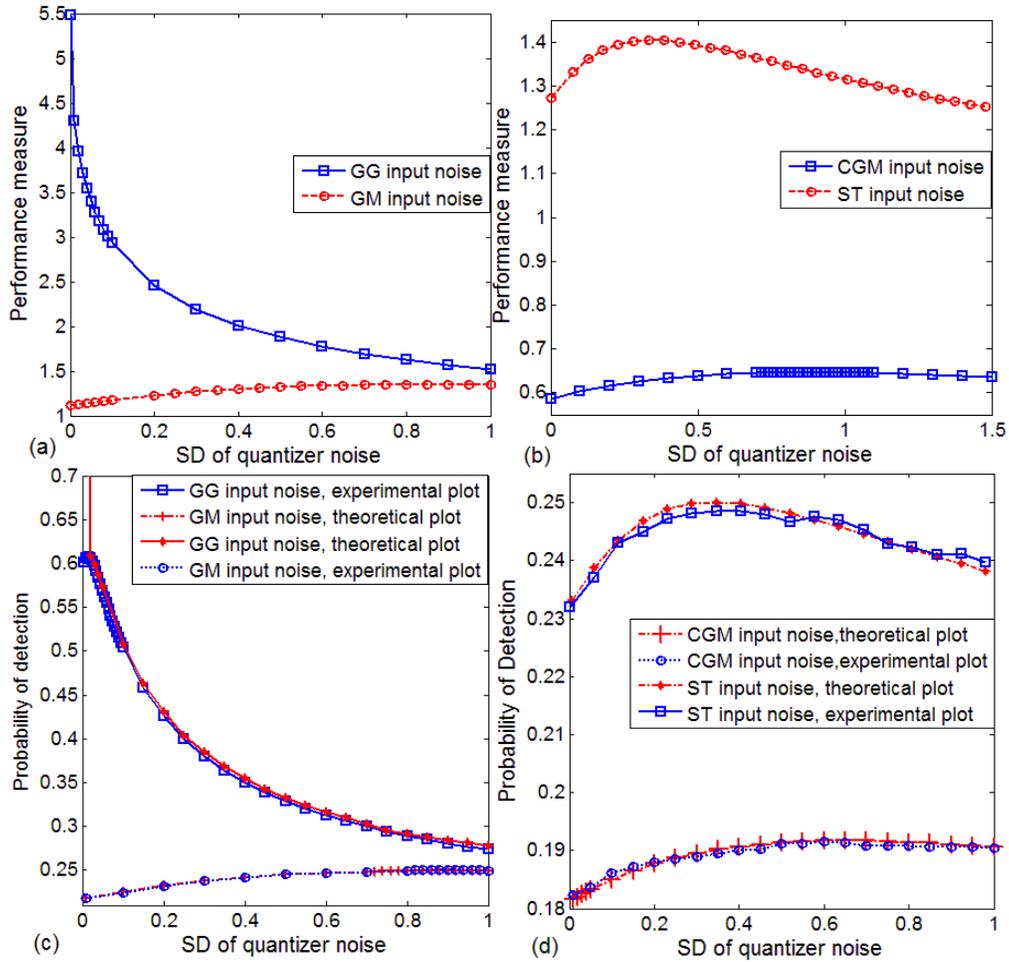


Fig. 4: Plots of  $G_{iq}$  vs.  $\sigma$  for (a) GG ( $p = 0.5$ ) and GM ( $u = 0.01$ ) input noise and (b) CGM ( $\alpha = 0.7$ ) and ST ( $v = 3$ ) input noise; and theoretical and experimental plots of  $P_D$  vs.  $\sigma$ , for (c) GG ( $p = 0.5$ ) and GM ( $u = 0.01$ ) input noise and (d) CGM ( $\alpha = 0.7$ ) and ST ( $v = 3$ ) input noise.  $P_D$  is plotted for  $P_F = 0.1$ ,  $N = 300$ .

and  $\Gamma(\cdot)$  is the Gamma function. The GG distribution is Gaussian for  $p = 2$  and leptokurtic for  $p < 2$ . Impulsiveness increases monotonically as  $p$  is decreased.

The pdf of a ST random variable with unit variance and degrees of freedom  $v$  is given by

$$f_{ST}(\xi, v) = \frac{\sqrt{(v+1)/2}}{\sqrt{(v-2)\pi} \sqrt{v/2}} \left(1 + \frac{\xi^2}{v-2}\right)^{-\left(\frac{v+1}{2}\right)}, v > 2 \quad (43)$$

For a finite value of  $v$ , the ST distribution represents a class of heavy-tailed leptokurtic distributions with algebraic tails. The distribution approaches the Gaussian distribution with unit variance as the parameter  $v \rightarrow \infty$ .

We consider the following family of CGM pdfs

$$f_{CGM}(\xi, \alpha) = \alpha \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\xi^2\right) + (1-\alpha) \frac{\sqrt{2}}{\pi(1+2\xi^2)}, 0 \leq \alpha \leq 1. \quad (44)$$

The pdf in (44) is an approximation to the pdf of the symmetric  $\alpha$ -stable ( $S\alpha S$ ) distribution with scale factor  $1/\sqrt{2}$  [34]. As the value of mixture parameter  $\alpha$  is decreased, the impulsiveness of the pdf increases. When  $\alpha = 1$ , the pdf is

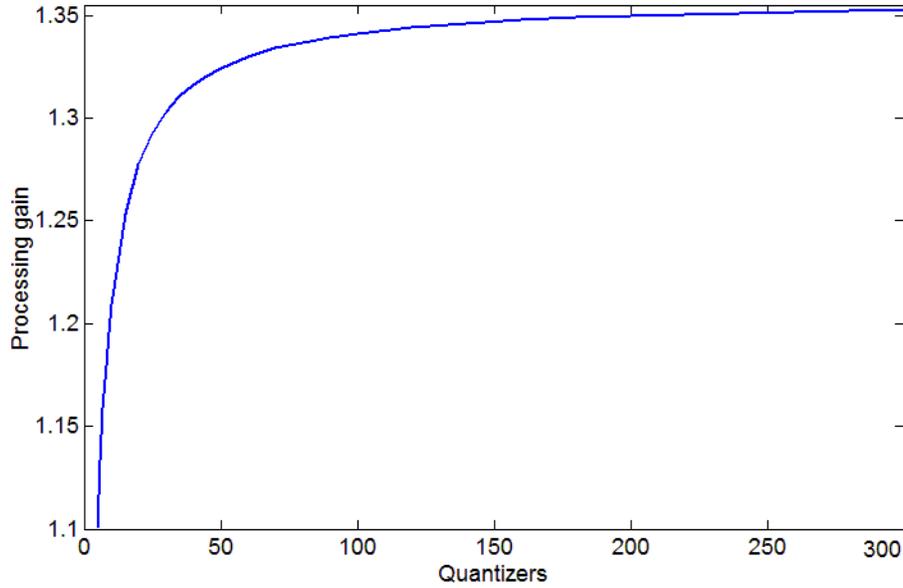


Fig. 5: Plot of  $G_{iq}$  vs. number of quantizers  $M$ . Input noise: GM ( $u = 0.01$ )

Gaussian with unit variance, and for  $\alpha < 1$ , the variance is infinite. Hence, for CGM input noise, the test statistic of the linear matched filter (LMF) also has infinite variance and it is not asymptotically Gaussian. However, the test statistics of the NQ and SSR detectors are asymptotically Gaussian with finite variance for CGM noise also. Consequently, in CGM noise, the quantities  $G_{iq}(\sigma)$  and  $G_{iq,SSR}$  continue to be the respective measures of performance of these detectors; but they cannot be regarded as processing gains of the corresponding preprocessors. Henceforth, the terms processing gain and preprocessor performance measure will be used interchangeably.

The four families of input noise pdfs mentioned above cover a wide range and variety of symmetric unimodal heavy-tailed distributions. These pdfs are widely used to model ambient noise in underwater acoustic applications [1,2,34].

### 5.2 Performance Analysis of SSR detector

The quantizer noise is assumed to be Gaussian throughout this subsection. All results and conclusions are similar for other choices of quantizer noise pdf. The issue of optimization of quantizer noise pdf is discussed in Section 6.

Figures 4(a) and 4(b) show the variations of the processing gain  $G_{iq}(\sigma)$  as the standard deviation  $\sigma$  of the quantizer noise is varied, for input noise with (a) GG and GM pdfs with unit variance, and (b) ST pdf with unit variance and CGM pdf respectively. Figures 4(c) and 4(d) show the corresponding plots of probability of detection  $P_D$  for probability of false alarm  $P_F = 0.1$ . The asymptotic theoretical  $P_D$  has been determined using (13), (26), (28), (20) and (21), and the experimental  $P_D$  has been determined by averaging over 35000 Monte Carlo simulations. It is seen that both  $G_{iq}$  and  $P_D$  are maximized at  $\sigma = 0.92$  when the input noise is GM ( $u = 0.01$ ), at  $\sigma = 0$  when the input noise is GG ( $p = 0.5$ ), at  $\sigma = 0.335$  when the input noise is ST ( $v = 3$ ), and at  $\sigma = 0.65$  when the input noise is CGM ( $\alpha = 0.7$ ). Hence, for weak signals, maximizing  $G_{iq}$  (and minimizing  $J_{iq}$ ) is equivalent to maximizing  $P_D$ . It is also seen from Fig. 4(c) and (d) that the theoretical and experimental plots of  $P_D$  are very close to each other. A slight deviation of the theoretical predictions from the experimental observations is observed only in the case of GG ( $p = 0.5$ ) input noise when the SD of quantizer noise is very close to zero. In this case, the theoretically determined optimal SD of quantizer noise is zero, and the theoretical probability of detection is greater than the observed one. However, this deviation does not exist for less leptokurtic GG input noise, for which the optimal value of SD of quantizer noise is greater than zero.

The relationship between the realizable processing gain  $G_{iq,SSR-M} = G_{iq,M}(\sigma_{opt})$  of the SSR preprocessor and the number of quantizers  $M$  is shown in Fig. 5 for GM ( $u = 0.01$ ) input noise. It is seen from Fig. 5 that  $G_{iq,SSR-M}$  increases monotonically and tends asymptotically to  $G_{iq,SSR}$  as  $M \rightarrow \infty$ . This is similar to the result obtained by Patel and Kosko

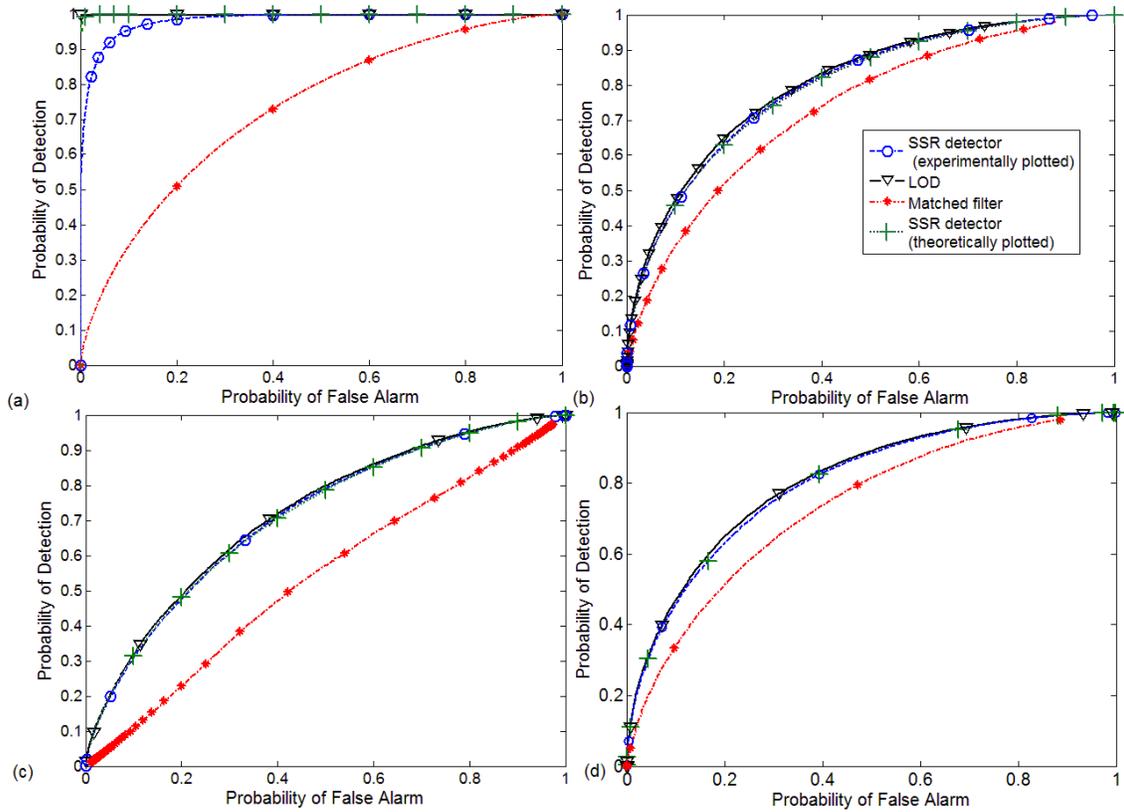


Fig. 6: ROCs of LO detector, SSR detector and matched filter. Input noise: (a) GG ( $p = 0.5$ ), (b) GM ( $u = 0.01$ ), (c) CGM ( $\alpha = 0.7$ ), (d) ST ( $v = 3$ ).  $A = 0.05$ ,  $N = 300$

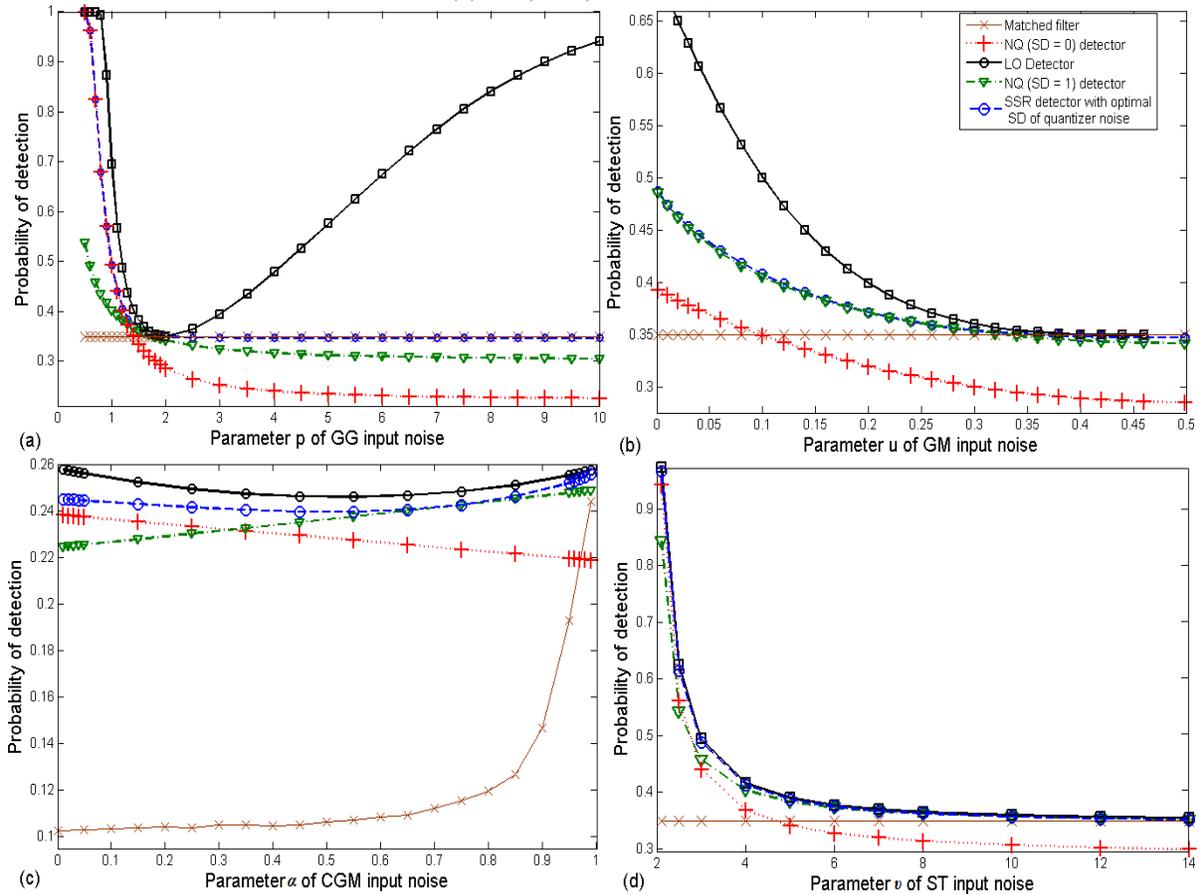


Fig. 7: Plots of  $P_D$  ( $P_F = 0.1$ ,  $N = 80$ ) vs. (a) parameter  $p$  of GG input noise, (b) parameter  $u$  of GM input noise, (c) parameter  $\alpha$  of CGM input noise and (d) parameter  $v$  of ST input noise, for 5 different detectors.

[23] that the rate of initial SR effect increases when the number of quantizers is increased. It is also seen that, as  $M$  is increased,  $G_{iq,SSR-M}$  rises quite rapidly towards its asymptotic value  $G_{iq-SSR}$ . Similar behavior is observed for other choices of input noise pdf also. Therefore, henceforth we will approximate  $G_{iq,SSR-M}$  by  $G_{iq-SSR}$  assuming that  $M$  is sufficiently large, and drop the distinction between the realizable and ideal processing gains.

In Figs. 6(a) – (d), the receiver operating characteristics (ROCs) of the LO detector, SSR detector and matched filter are plotted experimentally for GG, GM, CGM and ST input noise respectively. The theoretical ROCs for the SSR detector are also plotted, and they match very well with the experimental plots (except in the case of GG( $p = 0.5$ ) input noise, when the quantizer noise has SD = 0). These figures indicate that the performance of the SSR detector is nearly optimal and much better than that of the matched filter. The improvement in performance of the SSR detector with respect to the matched filter is particularly striking for CGM input noise.

We will now study the dependence of the performance of the SSR detector on the impulsiveness of input noise. At the outset, we recall that kurtosis, a measure of impulsiveness, increases when parameter  $p$  of the GG pdf, parameter  $u$  of the GM pdf, parameter  $\alpha$  of the CGM pdf or parameter  $v$  of the ST pdf is decreased. The variation of probability of detection  $P_D$  (for  $P_F = 0.1$ ) with respect to the input noise parameters  $p$ ,  $u$ ,  $\alpha$  and  $v$  is plotted in Figs. 7(a) - 7(d). Each panel contains five plots corresponding to five different detectors, viz. (1) LO, (2) SSR, (3) NQ with  $\sigma = 1$ , (4) NQ with  $\sigma = 0$  detectors, and (5) matched filter. Let the  $P_D$  of these detectors be denoted as  $P_{D,LO}$ ,  $P_{D,SSR}$ ,  $P_{D,NQ(1)}$ ,  $P_{D,NQ(0)}$  and  $P_{D,MF}$  respectively. All plots except the plot for the matched filter in Fig. 7(c) are theoretical. These figures provide the following observations and inferences.

(1) In GG, GM or ST input noise,  $P_{D,LO} \geq P_{D,MF}$ , and the equality holds if and only if  $p = 2$ ,  $u = 0.5$  or as  $v \rightarrow \infty$ . The implication is that LO detector reduces to matched filter if the input noise is Gaussian, and it performs better than matched filter in all other cases. This result is well-known.  $P_{D,LO}$  increases monotonically as  $p$ ,  $u$  or  $v$  is reduced (kurtosis of noise is increased);  $P_{D,LO} \rightarrow 1$  as  $p \rightarrow 0$ .

(2) For the SSR detector,  $P_{D,SSR}$  is slightly lower than  $P_{D,MF}$  for Gaussian input noise ( $p = 2$ ,  $u = 0.5$  or  $v \rightarrow \infty$ ). This conclusion is in conformity with the fact that the matched filter is the optimal detector in Gaussian noise.  $P_{D,SSR}$  increases monotonically as  $p$ ,  $u$ ,  $\alpha$  or  $v$  is decreased. Also,  $P_{D,SSR} = P_{D,MF}$  when  $p = 1.9$  (kurtosis = 0.1075),  $u = 0.36$  (kurtosis = 0.26) or  $v = 17$  (kurtosis = .4615). Hence, the SSR detector performs better than the matched filter in leptokurtic input noise irrespective of the shape of the noise pdf, unless the kurtosis is very close to 0; and the difference in performance between the two detectors increases monotonically as the kurtosis of input noise increases. Also  $P_{D,SSR} < P_{D,LO}$  for all  $p$ ,  $u$ ,  $v$  and  $\alpha$ . However, the asymptotic performance of SSR detector is fairly close to that of the LO detector in most cases.

(3) The plots for the NQ detector ( $\sigma = 1$ ) and NQ detector ( $\sigma = 0$ ) in Figs. 7(a)-7(d) illustrate the reduction in preprocessor performance due to non – optimal choice of  $\sigma$ . However, even with non – optimal values of  $\sigma$ , the performance of the NQ detectors in heavy-tailed input noise is much better than that of matched filter. We therefore conclude that quantization is the critical step in SSR preprocessing, while optimization of  $\sigma$  merely provides an incremental improvement in performance. This may be explained as follows. Quantization removes the outlier values from the impulsive input noise that contaminates the signal. But quantization also causes distortion of the signal which leads to a reduction of its correlation with the original signal. Addition of an optimal amount of quantizer noise helps in partially restoring the correlation and thus increasing the detector output when the signal is present. When the input noise becomes more impulsive, the benefit of SNR enhancement due to the quantization-induced removal of large noise values becomes more significant than the loss of correlation due to quantization-induced signal distortion, and therefore the contribution of quantizer noise to the enhancement of correlation also diminishes. It follows from the foregoing discussion that the SSR detector is fairly robust with respect to error in modeling the input noise pdf, since quantization is a more critical step as compared to fine – tuning the intensity of quantizer noise. Optimization of  $\sigma$  based on an approximate model of input noise pdf is therefore sufficient to ensure a performance that is much better than that of the matched filter.

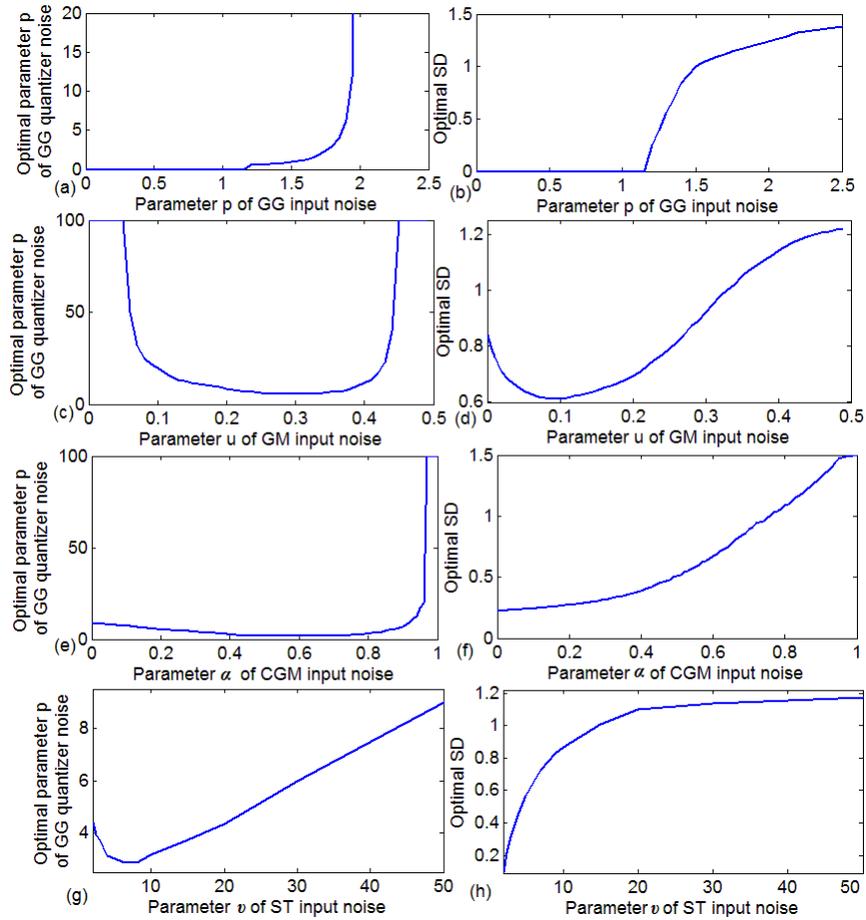


Fig. 8: Plots of (a) optimal  $p_q$  vs.  $p$  and (b)  $\sigma_{opt}$  vs.  $p$ , for GG input noise, (c) optimal  $p_q$  vs.  $u$  and (d)  $\sigma_{opt}$  vs.  $u$ , for GM input noise, (e) optimal  $p_q$  vs.  $\alpha$  and (f)  $\sigma_{opt}$  vs.  $\alpha$ , for CGM input noise, (g) optimal  $p_q$  vs.  $v$  and (h)  $\sigma_{opt}$  vs.  $v$ , for ST input noise. Quantizer noise is GG.

## 6. Optimization of quantizer noise pdf

The design of the optimal SSR detector for a given input noise pdf  $f_i(\zeta)$  requires the determination of the optimal standard deviation and the optimal pdf  $f_q(\zeta)$  of the quantizer noise. One way of obtaining the optimal pdf of quantizer noise is to parameterize the quantizer noise pdf and maximize the processor gain  $G_{iq}$  with respect to all the parameters of the quantizer noise, including the standard deviation. Here we consider a restricted version of this approach by modeling the quantizer noise by a GG pdf with parameter  $p_q > 0$ . This choice is suggested by the fact that the entire range of values of kurtosis  $[-1.2, \infty)$  is included in the class of GG distributions. Now, the problem of optimization of the detector reduces to the readily solvable problem of maximizing  $G_{iq}(\sigma, p_q)$  with respect to  $\sigma$  and  $p_q$ . For example, it can be shown that for CGM ( $\alpha = 0.7$ ), GM ( $u = 0.01$ ), ST ( $v = 3$ ) and GG ( $p = 0.5$ ) input noises, the optimal values of  $(\sigma, p_q)$  are  $(0.64, 2)$ ,  $(0.76, \infty)$ ,  $(0.31, 3.7)$ , and  $(0, -)$  respectively.

In Fig.8, the optimal values of  $\sigma$  and  $p_q$  are plotted against parameter  $p$  of GG input noise, parameter  $u$  of GM input noise, parameter  $\alpha$  of CGM input noise and parameter  $v$  of ST input noise respectively. Figures 8(a), (c), (e) and (g) indicate that the optimal value of  $p_q$  exhibits a wide range of variation with respect to input noise pdf. The optimal value of  $p_q \rightarrow \infty$  (uniform distribution) as the impulsiveness of the input noise pdf approaches that of Gaussian noise. Using a different optimality condition viz. maximal rate of initial SR effect, Patel and Kosko [23] have shown that uniform quantizer noise is optimal for a wide range of input noise pdfs.

Figures 8(b), 8(f) and 8(h) indicate that  $\sigma_{opt}$  decreases as impulsiveness of GG or CGM input noise is increased. This trend is in conformity with the observation of Kosko and Mitaim [13] that SR effect fades as noise becomes more impulsive. This decrease in  $\sigma_{opt}$  is because as the input noise becomes more impulsive, the benefit of

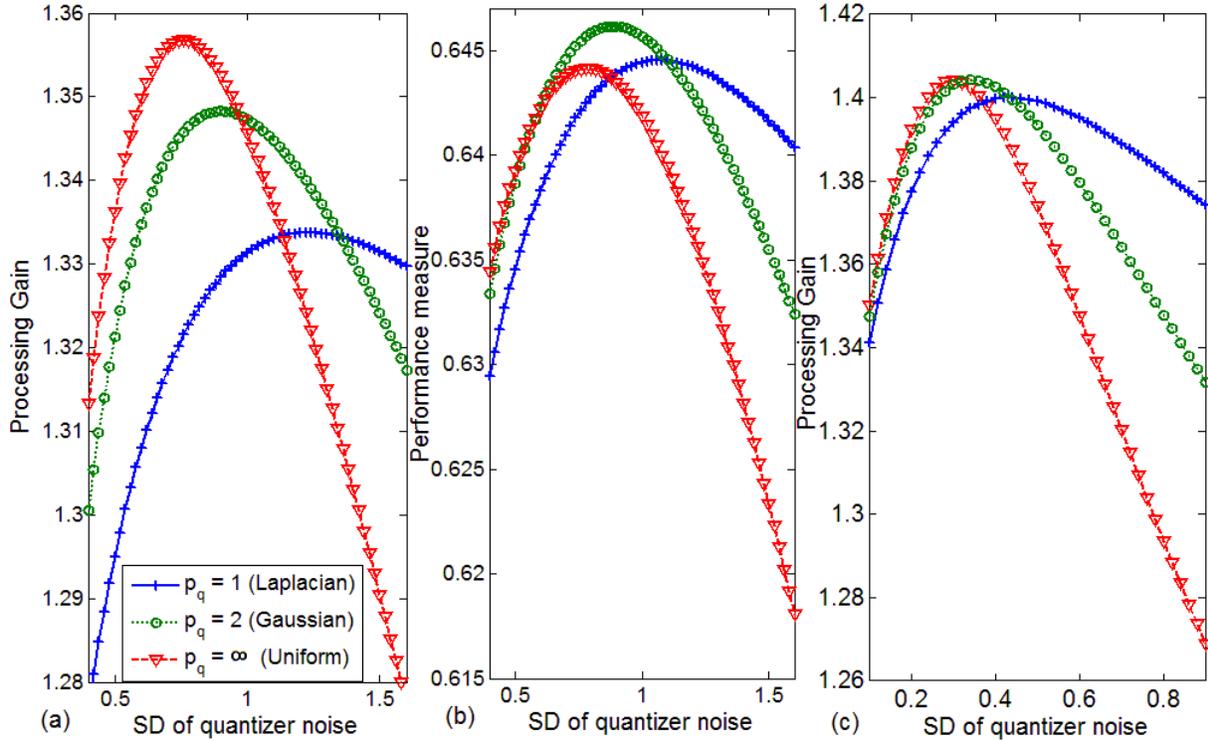


Fig. 9: Plots of  $G_{iq}$  vs.  $\sigma$  for different values of  $p_q$ , (a) GM ( $u = 0.01$ ), (b) CGM ( $\alpha = 0.7$ ) and (c) ST ( $\nu = 3$ ) input noise. Quantizer noise is GG.

quantization in removing large outlier values of noise becomes more significant compared to the signal distortion due to quantization. Therefore the importance of adding quantizer noise to enhance the signal quality diminishes as the impulsiveness of the noise increases. But the plot for GM input noise in Fig. 8(d) does not fully follow this trend of monotonic variation. In Fig.8(b) (GG input noise),  $\sigma_{opt} = 0$  for  $p < 1.05$ . It follows that, for GG input noise with  $p < 1.05$ , injection of additional noise does not provide improvement in performance over that of a simple quantizer-correlator.

Figure 9 shows plots of  $G_{iq}$  versus  $\sigma$  for different values of  $p_q$  for GM ( $u = 0.01$ ), CGM ( $\alpha = 0.7$ ) and ST ( $\nu = 3$ ) input noises. These plots indicate that the change in the peak value of  $G_{iq}$  due to variation in  $p_q$  is very small. Hence, the choice of quantizer noise pdf does not have a significant impact on the performance of SSR detector; a near-optimal performance may be obtained by choosing any value of  $p_q$  in the interval  $[2, \infty)$ . It is also seen that the peaks in Fig. 9 are quite broad, indicating low sensitivity to variation in the value of quantizer noise variance.

## 7. SSR detection of non-weak signals

The discussion so far has been confined to the problem of detection of weak signals with  $0 < A \ll 1$ . It is of interest to examine the performance of the SSR detector when the weak-signal condition is violated. This requires the determination of the means  $m_j$  and standard deviations  $\lambda_j$  of the test statistic  $T_{SSR}(\mathbf{x})$  without the weak-signal approximation, and then finding the value of  $\sigma$  that maximizes  $P_D$  for a given  $P_F$ . When  $A$  is not very small, the problem of optimization of the noisy quantizer detector gets modified due to the following reasons: (i) The approximation  $(\lambda_0/\lambda_1) = 1$  breaks down, (ii) values of  $\lambda_0/\lambda_1$  and the parameter  $d$  in (13) depend on both  $A$  and  $\sigma$ . Hence maximization of  $G_{iq,M}(\sigma)$  or  $G_{iq}(\sigma)$  does not lead to maximization of  $P_D$ . It is also evident from (13) that optimum  $\sigma$  depends not only on  $A$  but also on  $P_F$  in the case of non-weak signals.

The variation of optimum  $\sigma$  (for maximization of  $P_D$ ) with respect to  $A$  is shown in Fig. 10 for GG, GM, CGM and ST input noise. The optimum  $\sigma$  increases with  $A$  in an almost linear fashion if the input noise is CGM, GG or ST, but the variation is non-monotonic in the case of GM input noise. Plots of  $P_D$  (at  $P_F = 10^{-4}$ ) versus  $A$  for SSR

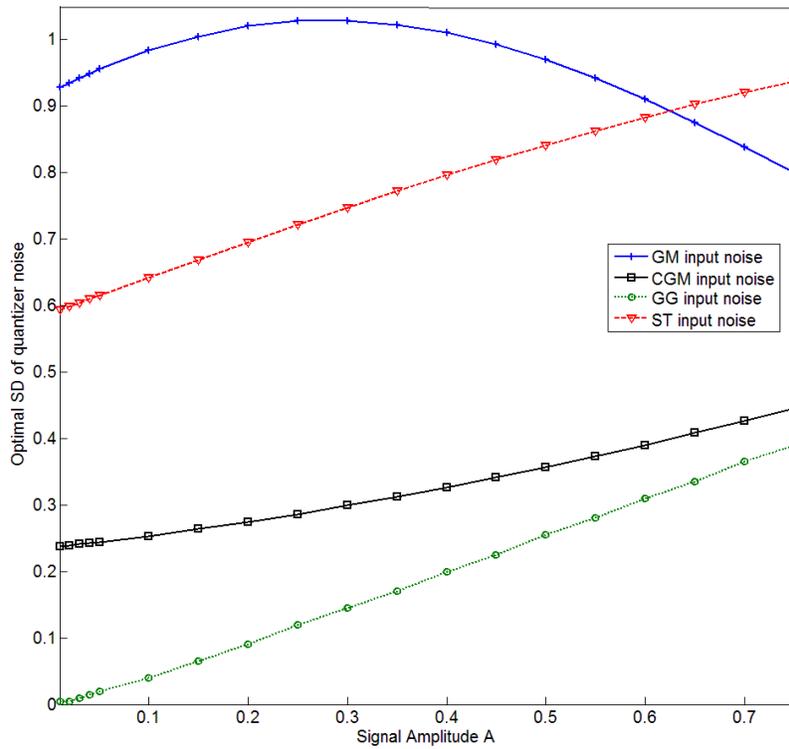


Fig. 10: Plots of  $\sigma_{opt}$  vs.  $A$  for GG ( $p = 0.5$ ), GM ( $u = 0.01$ ), CGM ( $\alpha = 0$ ) and ST ( $v = 3$ ) input noise.  $P_F = 10^{-4}$ ,  $N = 80$ .

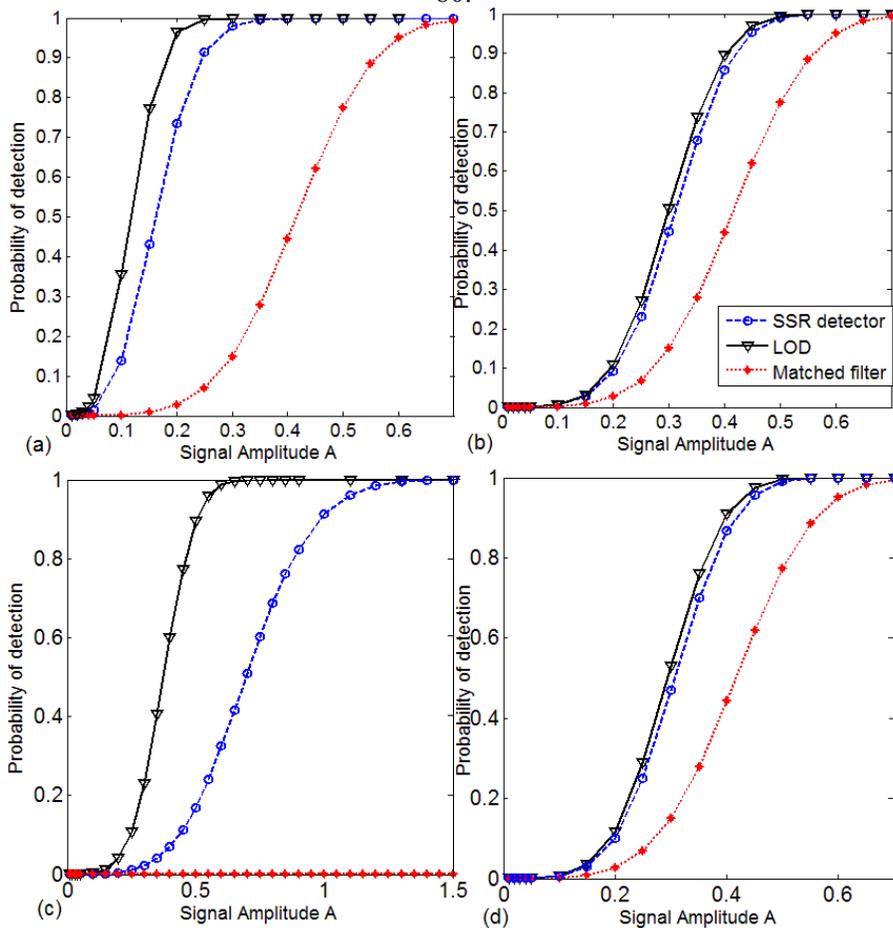


Fig. 11: Plots of  $P_D$  ( $P_F = 10^{-4}$ ,  $N = 80$ ) vs.  $A$  for SSR detector, OD and matched filter. Input noise: (a) GG ( $p = 0.5$ ), (b) GM ( $u = 0.01$ ), (c) CGM ( $\alpha = 0$ ), (d) ST ( $v = 3$ ).

detector, NP – optimal detector (OD) and matched filter are compared in Fig.11. The test statistic for OD is given by (1). The performance of the SSR detector is substantially superior to that of the matched filter and very close to that of the OD, indicating the SSR detector is effective for detection of non – weak signals as well.

## 8. Conclusions

This paper discusses the design and performance of a detector based on SSR, for detection of deterministic signals in non-Gaussian noise. The detector consists of a preprocessor and a matched filter. The preprocessor is an SSR system consisting of a parallel array of 2-level quantizers with additive i.i.d quantizer noise. The system is tuned to maximize the probability of detection by choosing the optimal value of standard deviation  $\sigma$  of the quantizer noise. In the case of weak signals, optimization of the preprocessor involves maximization of a function of  $\sigma$  that depends on the pdfs of input noise and quantizer noise. It is shown that maximizing the probability of detection also minimizes the mean-square difference between the output of the quantizer array and that of the optimum nonlinear transformation of the LO detector.

The performance of the SSR detector is illustrated by considering four families of input noise pdfs that cover a wide range of unimodal symmetric heavy-tailed noise distributions. It is shown that the SSR preprocessor provides a significant improvement in performance over the replica correlator or matched filter. For weak signals, the performance of the SSR detector is very close to that of the LO detector in most cases. The improvement stems primarily from quantization, while the addition of optimum amount of quantizer noise improves the performance further and brings it closer to that of the LO detector. For GG, ST and CGM input noise, the optimum value of  $\sigma$  decreases as the impulsiveness of input noise increases. Further improvement in performance is possible through an optimal choice of quantizer noise pdf. A parametric solution to this optimization problem is obtained by restricting the quantizer noise pdf to be generalized Gaussian (with parameter  $p_q$ ). It is shown that the optimal pdf of quantizer noise depends on the input noise pdf; but the sensitivity of the SSR detector performance to non-optimal choice of quantizer noise pdf is quite low. This implies that the SSR detector is robust with respect to errors in modeling the input noise pdf. Near-optimal performance can be achieved by choosing Gaussian or uniform quantizer noise pdf and tuning its standard deviation with an approximate model of input noise.

The procedure for the design of the SSR preprocessor can be readily extended to the detection of non-weak signals. In this case, the optimal value of  $\sigma$  depends on the signal amplitude and false alarm probability. The SSR detector provides a significant improvement in performance over the matched filter in the case of non-weak signals also.

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